Applied Mathematics & Optimization (2019) 80:835–852
https://doi.org/10.1007/s00245-019-09610-0

On Nonlocal Variational and Quasi-Variational Inequalities with Fractional Gradient

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Published online: 14 September 2019

Abstract
We extend classical results on variational inequalities with convex sets with gradient constraint to a new class of fractional partial differential equations in a bounded domain with constraint on the distributional Riesz fractional gradient, the σ-gradient ($0 < \sigma < 1$). We establish continuous dependence results with respect to the data, including the threshold of the fractional σ-gradient. Using these properties we give new results on the existence to a class of quasi-variational variational inequalities with fractional gradient constraint via compactness and via contraction arguments. Using the approximation of the solutions with a family of quasilinear penalisation problems we show the existence of generalised Lagrange multipliers for the σ-gradient constrained problem, extending previous results for the classical gradient case, i.e., with $\sigma = 1$.

Keywords Fractional gradient · Variational inequalities · Nonlocal quasi-variational inequalities · Gradient constraint · Lagrange multiplier

1 Introduction

In a series of two interesting papers [13] and [14], Shieh and Spector have considered a new class of fractional partial differential equations. Instead of using the well-known fractional Laplacian, their starting concept is the distributional Riesz fractional gradient of order $\sigma \in (0, 1)$, which will be called here the σ-gradient D^σ, for brevity: for $u \in L^p(\mathbb{R}^N)$, $1 < p < \infty$, we set

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$$\left(D^\sigma u\right)_j = \frac{\partial^\sigma u}{\partial x_j^\sigma} = \frac{\partial}{\partial x_j} I_{1-\sigma} u, \qquad 0 < \sigma < 1, \quad j = 1, \ldots, N, \tag{1.1}$$

where $\frac{\partial}{\partial x_j}$ is taken in the distributional sense, for every $v \in \mathscr{C}_0^\infty(\mathbb{R}^N)$,

$$\left\langle \frac{\partial^\sigma u}{\partial x_j^\sigma}, v \right\rangle = -\left\langle I_{1-\sigma} u, \frac{\partial v}{\partial x_j} \right\rangle = -\int_{\mathbb{R}^N} (I_{1-\sigma} u) \frac{\partial v}{\partial x_j}\, dx,$$

with I_α denoting the Riesz potential of order α, $0 < \alpha < 1$:

$$I_\alpha u(x) = (I_\alpha * u)(x) = \gamma_{N,\alpha} \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-\alpha}}\, dy, \qquad \text{with } \gamma_{N,\alpha} = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\pi^{\frac{N}{2}}\, 2^\alpha\, \Gamma\left(\frac{\alpha}{2}\right)}.$$

As it was shown in [13], D^σ has nice properties for $u \in \mathscr{C}_0^\infty(\mathbb{R}^N)$, namely

$$D^\sigma u \equiv D(I_{1-\sigma} u) = I_{1-\sigma} * Du, \tag{1.2}$$

$$(-\Delta)^\sigma u = -\sum_{j=1}^N \frac{\partial^\sigma}{\partial x_j^\sigma} \frac{\partial^\sigma}{\partial x_j^\sigma} u, \tag{1.3}$$

where the well-known fractional Laplacian may be given, for a suitable constant $C_{N,\sigma}$, by (see, for instance, [4]):

$$(-\Delta)^\sigma u \equiv C_{N,\sigma}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2\sigma}}\, dy.$$

It was also observed in [14] that the σ-gradient is an example of the non-local gradients considered in [9], which can be also given by

$$D^\sigma u(x) = R(-\Delta)^{\frac{\sigma}{2}} u(x) = (1-\sigma-N)\gamma_{N,1-\sigma} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+\sigma}} \frac{x-y}{|x-y|}\, dy, \tag{1.4}$$

in terms of the vector-valued Riesz transform (see [15], with $\rho_N = \Gamma\left(\frac{N+1}{2}\right)/\pi^{\frac{N+1}{2}}$):

$$Rf(x) = \rho_N\ \text{P.V.} \int_{\mathbb{R}^N} f(y) \frac{x-y}{|x-y|^{N+1}}\, dy.$$

We observe that, from the properties of D^σ and a result of [7] on the Riesz kernel as approximation of the identity as $\alpha \to 0$, the σ-gradient approaches the standard gradient as $\sigma \to 1$: if $Du \in L^p(\mathbb{R}^N)^N \cap L^q(\mathbb{R}^N)^N$, $1 < q < p$, then $D^\sigma u \underset{\sigma \to 1}{\longrightarrow} Du$ in $L^p(\mathbb{R}^N)^N$.

Introducing the vector space of fractional differentiable functions as the closure of $\mathscr{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\sigma,p}^p = \|u\|_{L^p(\mathbb{R}^N)}^p + \|D^\sigma u\|_{(L^p(R^N))^m}^p, \quad 0 < \sigma < 1,\ p > 1,$$

by [13, Theorem 1.7] it is exactly the Bessel potencial space $L^{\sigma,p}(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N)$, $0 \leq s < \sigma$, where $W^{s,p}(\mathbb{R}^N)$ denotes the usual fractional Sobolev space. In [13] the solvability of the fractional partial differential equations with variable coefficients and Dirichlet data was treated in the case $p = 2$, as well as the minimization of the integral functionals of the σ-gradient with p-growth, leading to the solvability of a fractional p-Laplace equation of a novel type.

In this work we are concerned with the Hilbertian case $p = 2$ in a bounded domain $\Omega \subset \mathbb{R}^N$, with Lipschitz boundary, where the homogeneous Dirichlet problem for a general linear PDE with measurable coefficients is considered under an additional constraint on the σ-gradient. We shall consider all solutions in the usual Sobolev space

$$H_0^\sigma(\Omega), \quad \text{with norm } \|u\|_{H_0^\sigma(\Omega)} = \|D^\sigma u\|_{L^2(\Omega)^N}, \quad 0 < \sigma < 1, \tag{1.5}$$

which, by the Sobolev-Poincaré inequality, is equivalent to the usual Hilbertian norm induced from $L^{\sigma,2}(\mathbb{R}^N) = W^{\sigma,2}(\mathbb{R}^N) = H^\sigma(\mathbb{R}^N)$, $0 < \sigma < 1$ in the closure of the Cauchy sequences of functions in $\mathscr{C}_0^\infty(\Omega)$ (see [13]).

For nonnegative functions $g \in L^\infty(\Omega)$, we consider the nonempty convex sets of the type

$$\mathbb{K}_g^\sigma = \left\{ v \in H_0^\sigma(\Omega) : |D^\sigma v| \leq g \text{ a.e. in } \Omega \right\}. \tag{1.6}$$

Let $f \in L^1(\Omega)$ and $A : \Omega \to \mathbb{R}^{N\times N}$ be a measurable, bounded and positive definite matrix. We shall consider, in Sect. 2, the well-posedness of the variational inequality

$$u \in \mathbb{K}_g^\sigma : \qquad \int_\Omega A D^\sigma u \cdot D^\sigma(v - u) \geq \int_\Omega f(v - u), \qquad \forall v \in \mathbb{K}_g^\sigma. \tag{1.7}$$

In particular, we obtain precise estimates for the continuous dependence of the solution u with respect to f and g, and so we extend well-known results for the classical case $\sigma = 1$ (see [12] and its references).

Extending the result of [2] for the gradient ($\sigma = 1$) case, we prove in Sect. 3 the existence of generalised Lagrange multipliers for the σ-gradient constrained problem. More precisely, we show the existence of $(\lambda, u) \in L^\infty(\Omega)' \times \Upsilon_\infty^\sigma(\Omega)$ such that

$$\langle \lambda D^\sigma u, D^\sigma v\rangle_{\left(L^\infty(\Omega)^N\right)' \times L^\infty(\Omega)^N} + \int_\Omega A D^\sigma u \cdot D^\sigma v = \int_\Omega f v, \qquad \forall v \in \Upsilon_\infty^\sigma(\Omega), \tag{1.8a}$$

$$|D^\sigma u| \leq g \ \text{ a.e. in } \Omega, \qquad \lambda \geq 0 \text{ and } \lambda(|D^\sigma u| - g) = 0 \ \text{ in } L^\infty(\Omega)' \tag{1.8b}$$

and, moreover, u solves (1.7).

Here, for each σ, we have set

$$\Upsilon_\infty^\sigma(\Omega) = \left\{ v \in H_0^\sigma(\Omega) : D^\sigma v \in L^\infty(\Omega)^N \right\}, \qquad 0 < \sigma < 1, \tag{1.9}$$

and

$$\langle \lambda \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{(L^\infty(\Omega)^N)' \times L^\infty(\Omega)^N} = \langle \lambda, \boldsymbol{\alpha} \cdot \boldsymbol{\beta} \rangle_{L^\infty(\Omega)' \times L^\infty(\Omega)}$$
$$\forall \lambda \in L^\infty(\Omega)' \; \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in L^\infty(\Omega)^N.$$

Finally, in the Sect. 4 we consider the solvability of solutions to quasi-variational inequalities corresponding to (1.7) when the threshold $g = G[u]$ and therefore also the convex set (1.6) depend on the solution $u \in \mathbb{K}^\sigma_{G[u]}$. We give sufficient conditions on the nonlinear and nonlocal operator $v \mapsto G[v]$ to obtain the existence of at least one solution u of (1.7) with $\mathbb{K}^\sigma_g$ replaced by $K^\sigma_{G[u]}$, by compactness methods, as in [6] for the case $\sigma = 1$. In a special case, when $G[u](x) = \Gamma(u)\varphi(x)$ is strictly positive and separates variables with a Lipschitz functional $\Gamma : L^2(\Omega) \to \mathbb{R}^+$, we adapt an idea of [5] (see also [12]) to obtain, by a contraction principle, the existence and uniqueness of the solution of the quasi-variational inequality under the "smallness" of the product of f with the Lipschitz constant of Γ and the inverse of its positive lower bound.

2 The Variational Inequality with σ-Gradient Constraint

For some $a_*, a^* > 0$, let $A = A(x) : \Omega \to \mathbb{R}^{N\times N}$ be a bounded and measurable matrix, not necessarily symmetric, such that, for a.e. $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}^N$:

$$a_*|\xi|^2 \le A(x)\xi \cdot \xi \le a^*|\xi|^2. \tag{2.1}$$

Fixed $\nu > 0$, we define

$$L^\infty_\nu(\Omega) = \big\{ v \in L^\infty(\Omega) : v(x) \ge \nu > 0 \text{ a.e. } x \in \Omega \big\}. \tag{2.2}$$

For any $g \in L^\infty_\nu(\Omega)$ it is clear that the convex set $\mathbb{K}^\sigma_g$ defined in (1.6) is non-empty, closed and, by Sobolev embeddings, we have, using the notation (1.9), for all $0 < \beta < \sigma$:

$$\mathbb{K}^\sigma_g \subset \Upsilon^\sigma_\infty(\Omega) \subset \mathscr{C}^{0,\beta}(\overline{\Omega}) \subset L^\infty(\Omega), \tag{2.3}$$

where $\mathscr{C}^{0,\beta}(\overline{\Omega})$ is the space of Hölder continuous functions with exponent β. Indeed, we recall (see for instance [3]) the embedding for the fractional Sobolev spaces $0 < \sigma \le 1$, $1 < p < \infty$:

$$W^{\sigma,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for every } q \le \tfrac{Np}{N-\sigma p}, \quad \text{if } \sigma p < N, \tag{2.4a}$$
$$W^{\sigma,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for every } q < \infty, \quad \text{if } \sigma p = N, \tag{2.4b}$$
$$W^{\sigma,p}(\Omega) \hookrightarrow L^\infty(\Omega) \cap \mathscr{C}^{0,\beta}(\overline{\Omega}), \quad \text{for every } 0 < \beta \le \sigma - \tfrac{N}{p}, \quad \text{if } \sigma p > N, \tag{2.4c}$$

with continuous embeddings, which are also compact if $q < \frac{Np}{N-\sigma p}$ in (2.4a) and $\beta < \sigma - \frac{N}{p}$ in (2.4c). In particular, we have

$$H_0^\sigma(\Omega) \hookrightarrow L^{2^*}(\Omega) \quad \text{and} \quad L^{2^\#}(\Omega) \hookrightarrow H^{-\sigma}(\Omega) = \left(H_0^\sigma(\Omega)\right)', \ 0 < \sigma < 1, \tag{2.5}$$

where we set $2^* = \frac{2N}{N-2\sigma}$ and $2^\# = \frac{2N}{N+2\sigma}$ when $\sigma < \frac{N}{2}$, and if $N = 1$ we denote $2^* = q$, $2^\# = q' = \frac{q}{q-1}$ when $\sigma = \frac{1}{2}$ and $2^* = \infty$, $2^\# = 1$ when $\sigma > \frac{1}{2}$.

Here we are also assuming that $\Omega \subset \mathbb{R}^N$ is an open, bounded domain with Lipschitz boundary, and we may conclude (2.3) from (2.4a)–(2.4c) by using a bootstrap argument.

Therefore, in the right hand side of the variational inequality (1.7), for $g_i \in L^\infty(\Omega)$, we can take $f_i \in L^1(\Omega)$, and the first two theorems give continuous dependence results with precise estimates for two different problems with $i = 1, 2$:

$$u_i \in \mathbb{K}^\sigma_{g_i} : \qquad \int_\Omega A D^\sigma u_i \cdot D^\sigma(v - u_i) \geq \int_\Omega f_i(v - u_i), \quad \forall v \in \mathbb{K}^\sigma_{g_i}. \tag*{$(2.15)_i$}$$

Theorem 2.1 *Under the assumptions (2.1), for each $f_i \in L^1(\Omega)$ and each $g_i \in L^\infty(\Omega)$, $g_i \geq 0$, there exists a unique solution u_i to $(2.15)_i$ such that*

$$u_i \in \mathbb{K}^\sigma_{g_i} \cap \mathscr{C}^{0,\beta}(\overline{\Omega}), \quad \textit{for all } 0 < \beta < \sigma. \tag{2.16}$$

When $g_1 = g_2$, the solution map $L^1(\Omega) \ni f \mapsto u \in H_0^\sigma(\Omega)$ is $\frac{1}{2}$-Hölder continuous, i.e., for some $C_1 > 0$, we have

$$\|u_1 - u_2\|_{H_0^\sigma(\Omega)} \leq C_1 \|f_1 - f_2\|^{\frac{1}{2}}_{L^1(\Omega)}. \tag{2.17}$$

Moreover, if in addition $f_i \in L^{2^\#}(\Omega)$, $2^\#$ defined in (2.5), $i = 1, 2$ and $g_1 = g_2$, then $L^{2^\#}(\Omega) \ni f \mapsto u \in H_0^\sigma(\Omega)$ is Lipschitz continuous:

$$\|u_1 - u_2\|_{H_0^\sigma(\Omega)} \leq C_\# \|f_1 - f_2\|_{L^{2^\#}(\Omega)}, \tag{2.18}$$

for $C_\# = C_*/a_* > 0$, where C_* is the constant of the Sobolev embedding $H_0^\sigma(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

Proof Suppose that $f_i \in L^{2^\#}(\Omega) \subset H^{-\sigma}(\Omega)$. Since the assumption (2.1) implies that A defines a continuous bilinear and coercive form over $H_0^\sigma(\Omega)$, the existence and uniqueness of the solution $u_i \in \mathbb{K}^\sigma_i$ to $(2.15)_i$ is an immediate consequence of the Stampacchia Theorem (see, for instance, [11, p. 95]), and (2.16) follows from (2.3).

With our notation (1.5), the estimate (2.18) follows easily from $(2.15)_i$ with $g_1 = g_2$ and $v = u_j$ $(i, j = 1, 2, i \neq j)$ from

$$a_* \|\overline{u}\|^2_{H_0^\sigma(\Omega)} \leq \int_\Omega A D^\sigma \overline{u} \cdot D^\sigma \overline{u} \leq \|\overline{f}\|_{L^{2^\#}(\Omega)} \|\overline{u}\|_{L^{2^*}(\Omega)} \leq C_* \|\overline{f}\|_{L^{2^\#}(\Omega)} \|\overline{u}\|_{H_0^\sigma(\Omega)},$$

where we have set $\overline{u} = u_1 - u_2$ and $\overline{f} = f_1 - f_2$.

By (2.3), letting κ be such that

$$\|v\|_{L^\infty(\Omega)} \leq \kappa, \qquad \forall v \in \mathbb{K}^\sigma_{g_1}, \tag{2.19}$$

we may easily conclude the estimate (2.17) with $C_1 = \sqrt{2\kappa/a_*}$ for $f_1, f_2 \in L^{2^\#}(\Omega) \subset L^1(\Omega)$ from $(1.5)_i$ and

$$a_*\|\overline{u}\|^2_{H^\sigma_0(\Omega)} \leq \|\overline{f}\|_{L^1(\Omega)}\|\overline{u}\|_{L^\infty(\Omega)} \leq 2\kappa\|\overline{f}\|_{L^1(\Omega)}.$$

Finally, the solvability of $(2.15)_i$ for f_i only in $L^1(\Omega)$ can be easily obtained by taking an approximating sequence of $f_i^n \in L^{2^\#}(\Omega)$ such that $f_i^n \underset{n}{\to} f_i$ in $L^1(\Omega)$ and using (2.17) for that (Cauchy) sequence. The proof is complete. □

Remark 2.1 As in [13] it is possible to extend the variational inequality with σ-gradient to arbitrary open domains $\Omega \subset \mathbb{R}^N$ with a generalised Dirichlet data $\varphi \in H^\sigma(\mathbb{R}^N)$ such that $I_{1-\sigma} * \varphi$ is well-defined and $D^\sigma\varphi \in L^\infty(\mathbb{R}^N)$. This would require in the definition (1.6) of $\mathbb{K}^\sigma_g$ to replace $H^\sigma_0(\Omega)$ by the space

$$H^\sigma_\varphi = \left\{v \in H^\sigma(\mathbb{R}^N) : v = \varphi \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}$$

and, in addition, technical compatibility assumptions on φ and g to guarantee that the new $\mathbb{K}^\sigma_g \neq \emptyset$.

Remark 2.2 It is well-known that if, in addition, A is symmetric, i.e. $A = A^T$, the variational inequality (1.7) corresponds (and is equivalent) to the optimisation problem (see, for instance, [11])

$$u \in \mathbb{K}^\sigma_g : \qquad \mathcal{J}(u) \leq \mathcal{J}(v), \qquad \forall v \in \mathbb{K}^\sigma_g,$$

where $\mathcal{J} : \mathbb{K}^\sigma_g \to \mathbb{R}$ is the convex functional

$$\mathcal{J}(v) = \frac{1}{2}\int_\Omega AD^\sigma v \cdot D^\sigma v - \int_\Omega fv.$$

Theorem 2.2 *Under the framework of the previous theorem, when $f_1 = f_2 \in L^1(\Omega)$, the solution map*

$$L^\infty_\nu(\Omega) \ni g \mapsto u \in H^\sigma_0(\Omega)$$

is also $\frac{1}{2}$-Hölder continuous, i.e., there exists $C_\nu > 0$ such that

$$\|u_1 - u_2\|_{H^\sigma_0(\Omega)} \leq C_\nu\|g_1 - g_2\|^{\frac{1}{2}}_{L^\infty(\Omega)}. \tag{2.20}$$

Proof Let $\eta = \|g_1 - g_2\|_{L^\infty(\Omega)}$ and, for $i, j = 1, 2$, $i \neq j$, notice that

$$u_{i_j} = \frac{\nu}{\nu + \eta} u_i \in \mathbb{K}^\sigma_{g_j},$$

if u_i denotes the unique solution of $(2.15)_i$ to g_i and f_i.

Denote by $\kappa = \max_{i=1,2}\{\|g_i\|_{L^\infty(\Omega)}, \|u_i\|_{L^\infty(\Omega)}\}$ and observe that for $i = 1, 2$,

$$|u_i - u_{i_j}| + |D^\sigma(u_i - u_{i_j})| \leq \frac{\eta}{\nu + \eta}\big(|u_i| + |D^\sigma u_i|\big) \leq 2\kappa\,\frac{\eta}{\nu}.$$

Hence, letting $v = u_{i_j}$ in $(2.15)_j$ and using (2.1) we get

$$\begin{aligned}
a_*\|u_1 - u_2\|^2_{H^\sigma_0(\Omega)} &\leq \int_\Omega AD^\sigma(u_1 - u_2)\cdot D^\sigma(u_1 - u_2)\\
&\leq \int_\Omega AD^\sigma u_1 \cdot D^\sigma(u_{2_1} - u_2) + \int_\Omega AD^\sigma u_2 \cdot D^\sigma(u_{1_2} - u_1)\\
&\quad + \int_\Omega f\big((u_1 - u_{1_2}) + (u_2 - u_{2_1})\big)\\
&\leq 2\kappa\,\frac{\eta}{\nu}\big(M\|g_1\|_{L^1(\Omega)} + M\|g_2\|_{L^1(\Omega)} + 2\|f\|_{L^1(\Omega)}\big) = C^2_\nu\|g_1 - g_2\|_{L^\infty(\Omega)},
\end{aligned}$$

with $C_\nu = \sqrt{2\kappa\big(M\|g_1\|_{L^1(\Omega)} + M\|g_2\|_{L^1(\Omega)} + 2\|f\|_{L^1(\Omega)}\big)/a_*\nu} > 0$,where $M = \|A\|_{L^\infty(\Omega)^{N^2}}$ which yields (2.20). $\square$

Remark 2.3 Using the trick of the above proof, if $g_n \underset{n}{\rightarrow} g$ in $L^\infty(\Omega)$ for a sequence $g_n \in L^\infty_\nu(\Omega)$, it is clear that, for any $w \in \mathbb{K}^\sigma_g$ we can choose $w_n \in \mathbb{K}^\sigma_{g_n}$ such that $w_n \underset{n}{\rightarrow} w$ in $H^\sigma_0(\Omega)$. On the other hand, also for any sequence $w_n \underset{n}{\rightharpoonup} w$ in $H^\sigma_0(\Omega)$-weak, with each $w_n \in \mathbb{K}^\sigma_{g_n}$, $g_n \underset{n}{\rightarrow} g$ in $L^\infty(\Omega)$ implies that also $w \in \mathbb{K}^\sigma_g$. These two conditions determine that if $g_n \underset{n}{\rightarrow} g$ in $L^\infty_\nu(\Omega)$ then the respective convex sets $\mathbb{K}^\sigma_{g_n}$ converge in the Mosco sense to $\mathbb{K}^\sigma_g$. An open question is to extend this convergence to the case $0 < \sigma < 1$, by dropping the strict positivity condition on g_n and g, as in [1] for $\sigma = 1$.

3 Existence of Lagrange Multipliers

In this section we prove the existence of solution of the problem (1.8a)–(1.8b).

For $\varepsilon \in (0, 1)$ and denoting $\widehat{k_\varepsilon} = \widehat{k_\varepsilon}(D^\sigma u^\varepsilon) = k_\varepsilon(|D^\sigma u^\varepsilon| - g)$ for simplicity, we define a family of approximated quasi-linear problems

$$\int_\Omega \big(\widehat{k_\varepsilon}(D^\sigma u^\varepsilon)D^\sigma u^\varepsilon + AD^\sigma u^\varepsilon\big)\cdot D^\sigma v = \int_\Omega f v \qquad \forall v \in H^\sigma_0(\Omega) \tag{3.1}$$

where $k_\varepsilon : \mathbb{R} \to \mathbb{R}$ is defined by

$$k_\varepsilon(s) = 0 \text{ for } s < 0, \quad k_\varepsilon(s) = e^{\frac{s}{\varepsilon}} - 1 \text{ for } 0 \le s \le \tfrac{1}{\varepsilon} \qquad k_\varepsilon(s) = e^{\frac{1}{\varepsilon^2}} - 1 \text{ for } s > \tfrac{1}{\varepsilon}.$$

Proposition 3.1 *Suppose that $g \in L^\infty_\nu(\Omega)$, $f \in L^{2^\#}(\Omega)$ and $A : \Omega \to \mathbb{R}^{N\times N}$ is a measurable, bounded and positive definite matrix. Then the quasi-linear problem (3.1) has a unique solution $u^\varepsilon \in H^\sigma_0(\Omega)$.*

Proof The operator $B_\varepsilon : H^\sigma_0(\Omega) \to H^{-\sigma}(\Omega)$ defined by

$$\langle B_\varepsilon v, w\rangle = \int_\Omega \left(\widehat{k_\varepsilon}(D^\sigma v)D^\sigma v + AD^\sigma v\right) \cdot D^\sigma w$$

is bounded, strongly monotone, coercive and hemicontinuous, so problem (3.1) has a unique solution (see, for instance, [8]). □

Lemma 3.1 *If $g \in L^\infty_\nu(\Omega)$, $f \in L^{2^\#}(\Omega)$, $A : \Omega \to \mathbb{R}^{N\times N}$ is a measurable, bounded and positive definite matrix and $1 \le q < \infty$, there exist positive constants C and C_q such that, for $0 < \varepsilon < 1$, setting $\widehat{k_\varepsilon} = k_\varepsilon(|D^\sigma u^\varepsilon| - g)$, the solution u^ε of the approximated problem (3.1) satisfies*

$$\|\widehat{k_\varepsilon}|D^\sigma u^\varepsilon|^2\|_{L^1(\Omega)} \le C, \tag{3.2a}$$

$$\|\widehat{k_\varepsilon}\|_{L^1(\Omega)} \le C, \tag{3.2b}$$

$$\|\widehat{k_\varepsilon} D^\sigma u^\varepsilon\|_{(L^\infty(\Omega)^N)'} \le C, \tag{3.2c}$$

$$\|\widehat{k_\varepsilon}\|_{L^\infty(\Omega)'} \le C \tag{3.2d}$$

$$\|D^\sigma u^\varepsilon\|_{L^q(\Omega^N)} \le C_q. \tag{3.2e}$$

Proof Using u^ε as test function in (3.1), we get

$$\begin{aligned}\int_\Omega \left(\widehat{k_\varepsilon} + a_*\right)|D^\sigma u^\varepsilon|^2 &\le \int_\Omega \widehat{k_\varepsilon}|D^\sigma u^\varepsilon|^2 + AD^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon \\ &= \int_\Omega f u^\varepsilon \le \frac{C_\#^2}{2a_*}\|f\|^2_{L^{2^\#}(\Omega)} + \frac{a_*}{2}\|D^\sigma u^\varepsilon\|^2_{L^2(\Omega)^N},\end{aligned}$$

since $A\xi \cdot \xi \ge a_*|\xi|^2$ for any $\xi \in \mathbb{R}^N$ by the assumptions on A. But $\widehat{k_\varepsilon} \ge 0$ and so

$$\frac{a_*}{2}\int_\Omega |D^\sigma u^\varepsilon|^2 \le \frac{C_\#^2}{2a_*}\|f\|^2_{L^{2^\#}(\Omega)},$$

concluding then (3.2a).

Observing that the function $\varphi_\varepsilon = \widehat{k_\varepsilon}\,(t^2 - g^2) + g^2\widehat{k_\varepsilon} \ge \nu^2\widehat{k_\varepsilon}$ and using (3.2a), there exists a positive constant C independent of ε such that

$$\nu^2\int_\Omega \widehat{k_\varepsilon} \le C.$$

This implies the uniform boundedness of $\widehat{k_\varepsilon}$ in $L^1(\Omega)$ and also in $L^\infty(\Omega)'$, i.e., (3.2b) and (3.2d) respectively.

To prove (3.2c), it is enough to notice that, for $\boldsymbol{\beta} \in L^\infty(\Omega)^N$,

$$\begin{aligned}\|\widehat{k_\varepsilon} D^\sigma u^\varepsilon\|_{(L^\infty(\Omega)^N)'} &= \sup_{\boldsymbol{\beta}\in L^\infty(\Omega)^N} \int_\Omega \widehat{k_\varepsilon} D^\sigma u^\varepsilon \cdot \boldsymbol{\beta} \\ &\leq \Big(\int_\Omega \widehat{k_\varepsilon} |D^\sigma u^\varepsilon|^2\Big)^{\frac{1}{2}} \Big(\int_\Omega \widehat{k_\varepsilon} |\boldsymbol{\beta}|^2\Big)^{\frac{1}{2}} \\ &\leq C\|\boldsymbol{\beta}\|_{L^\infty(\Omega)^N}.\end{aligned}$$

Because for $t-g>0$ we have $k_\varepsilon(t-g) \geq \frac{1}{m!}(t-g)^m$, for any $m\in\mathbb{N}$, then using (3.2b) we conclude (3.2e),(for details see, for instance [10]). □

Proposition 3.2 *For $g \in L^\infty_\nu(\Omega)$, $f \in L^{2^\#}(\Omega)$ and $A : \Omega \to \mathbb{R}^{N\times N}$ a measurable, bounded and positive definite matrix, the family $\{u^\varepsilon\}_\varepsilon$ of solutions of the approximated problems (3.1) converges weakly in $H^\sigma_0(\Omega)$ to the solution of the variational inequality (1.7).*

Proof The uniform boundedness of $\{u^\varepsilon\}_\varepsilon$ in $H^\sigma_0(\Omega)$ implies that, at least for a subsequence,

$$u^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u \quad \text{in } H^\sigma_0(\Omega). \tag{3.3}$$

For $v\in\mathbb{K}^\sigma_g$ we have, since $\widehat{k_\varepsilon}>0$ when $|D^\sigma u_\varepsilon| > g \geq |D^\sigma v|$,

$$\widehat{k_\varepsilon} D^\sigma u^\varepsilon \cdot D^\sigma(v-u^\varepsilon) \leq \widehat{k_\varepsilon}|D^\sigma u^\varepsilon|(|D^\sigma v| - |D^\sigma u^\varepsilon|) \leq 0$$

and so, testing the first equation of (3.1) with $v-u^\varepsilon$, we get

$$\int_\Omega AD^\sigma u^\varepsilon \cdot D^\sigma(v-u^\varepsilon) \geq \int_\Omega f(v-u^\varepsilon).$$

But

$$\begin{aligned}\int_\Omega AD^\sigma u^\varepsilon \cdot D^\sigma(v-u^\varepsilon) &= \int_\Omega AD^\sigma(u^\varepsilon - v)\cdot D^\sigma(v-u^\varepsilon) \\ &\quad + \int_\Omega AD^\sigma v \cdot D^\sigma(v-u^\varepsilon) \\ &\leq \int_\Omega AD^\sigma v\cdot D^\sigma(v-u^\varepsilon)\end{aligned}$$

So, utilizing the weak convergence $u^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u$ in $H^\sigma_0(\Omega)$,

$$\int_\Omega AD^\sigma v\cdot D^\sigma(v-u) \geq \int_\Omega f(v-u).$$

Let $w \in \mathbb{K}_g^\sigma$ and setting $v = u + \theta(w-u)$, then $v \in \mathbb{K}_g^\sigma$ for any $\theta \in (0,1]$ and we get

$$\theta \int_\Omega AD^\sigma(u+\theta(w-u)) \cdot D^\sigma(w-u) \geq \theta \int_\Omega f(w-u).$$

Dividing this inequality by θ and letting $\theta \to 0$, we obtain (1.7). The proof is concluded if we show that $u \in \mathbb{K}_g^\sigma$. Indeed we split Ω in three subsets

$$U_\varepsilon = \left\{|D^\sigma u^\varepsilon| - g \leq \sqrt{\varepsilon}\right\}, \quad V_\varepsilon = \left\{\sqrt{\varepsilon} \leq |D^\sigma u^\varepsilon| - g \leq \tfrac{1}{\varepsilon}\right\},$$
$$W_\varepsilon = \left\{|D^\sigma u^\varepsilon| - g > \tfrac{1}{\varepsilon}\right\}$$

and, following the steps in [10], we conclude that

$$\begin{aligned}
\int_\Omega \left(|D^\sigma u| - g\right)^+ &\leq \varliminf_{\varepsilon\to 0} \int_\Omega \left(\left(|D^\sigma u^\varepsilon| - g\right) \vee 0\right) \wedge \frac{1}{\varepsilon} \\
&= \varliminf_{\varepsilon\to 0} \left(\int_{U_\varepsilon} \left(|D^\sigma u^\varepsilon| - g\right) \vee 0 + \int_{V_\varepsilon} \left(|D^\sigma u^\varepsilon| - g\right) + \int_{W_\varepsilon} \tfrac{1}{\varepsilon}\right) \\
&\leq \varliminf_{\varepsilon\to 0} \left(\sqrt{\varepsilon}|\Omega| + \||D^\sigma u^\varepsilon| - g\|_{L^2(\Omega)} |V_\varepsilon|^{\frac{1}{2}} + \int_{W_\varepsilon} \tfrac{1}{\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0,
\end{aligned}$$

because

$$|V_\varepsilon| \leq \int_{V_\varepsilon} \frac{\widehat{k}_\varepsilon + 1}{e^{\frac{1}{\sqrt{\varepsilon}}}} \leq Ce^{\frac{-1}{\sqrt{\varepsilon}}} \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{and} \quad \int_{W_\varepsilon} \tfrac{1}{\varepsilon} = \tfrac{1}{\varepsilon} \int_{W_\varepsilon} \frac{\widehat{k}_\varepsilon + 1}{e^{\frac{1}{\varepsilon^2}}} \leq \frac{C}{\varepsilon} e^{-\frac{1}{\varepsilon^2}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

So $|D^\sigma u| \leq g$ a.e. in Ω, which means that $u \in \mathbb{K}_g^\sigma$.

The uniqueness of solution of the variational inequality (1.7) implies that the whole sequence $\{u^\varepsilon\}_\varepsilon$ converges to u in $H_0^\sigma(\Omega)$. □

Theorem 3.1 *If $g \in L_\nu^\infty(\Omega)$, $f \in L^{2^\#}(\Omega)$ and $A : \Omega \to \mathbb{R}^{N\times N}$ is a measurable, bounded and positive definite matrix, then problem* (1.8a)–(1.8b) *has a solution*

$$(\lambda, u) \in L^\infty(\Omega)' \times \Upsilon_\infty^\sigma(\Omega).$$

Proof By estimates (3.2c) and (3.2d) and the Banach–Alaoglu–Bourbaki theorem we have, at least for a subsequence,

$$\widehat{k}_\varepsilon D^\sigma u^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \Lambda \text{ weak in } \left(L^\infty(\Omega)^N\right)'$$

and

$$\widehat{k}_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \lambda \text{ weak in } L^\infty(\Omega)'.$$

For $v \in H_0^\sigma(\Omega)$, since

$$\int_\Omega \left(\widehat{k_\varepsilon} D^\sigma u^\varepsilon + AD^\sigma u^\varepsilon\right) \cdot D^\sigma v = \int_\Omega f v, \tag{3.4}$$

we obtain, letting $\varepsilon \to 0$ with $v \in \Upsilon_\infty^\sigma(\Omega)$,

$$\langle \Lambda, D^\sigma v\rangle + \int_\Omega AD^\sigma u \cdot D^\sigma v = \int_\Omega f v. \tag{3.5}$$

Taking $v = u^\varepsilon$ in (3.4) we get

$$\int_\Omega \widehat{k_\varepsilon} |D^\sigma u^\varepsilon|^2 + \int_\Omega AD^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon = \int_\Omega f u^\varepsilon \tag{3.6}$$

Observe first that

$$\begin{aligned}\int_\Omega AD^\sigma (u^\varepsilon - u) \cdot D^\sigma u^\varepsilon &= \int_\Omega AD^\sigma (u^\varepsilon - u) \cdot D^\sigma (u^\varepsilon - u)\\ &+ \int_\Omega AD^\sigma (u^\varepsilon - u) \cdot D^\sigma u \geq \int_\Omega AD^\sigma (u^\varepsilon - u) \cdot D^\sigma u \end{aligned}\tag{3.7}$$

and therefore

$$\int_\Omega AD^\sigma u \cdot D^\sigma u \leq \varliminf_{\varepsilon\to 0} \int_\Omega AD^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon.$$

So, from(3.6) and (3.5) with $v = u$,

$$\begin{aligned}\varliminf_{\varepsilon\to 0} \int_\Omega \widehat{k_\varepsilon} |D^\sigma u^\varepsilon|^2 + \int_\Omega AD^\sigma u \cdot D^\sigma u &\leq \varliminf_{\varepsilon\to 0} \left(\int_\Omega \widehat{k_\varepsilon} |D^\sigma u^\varepsilon|^2 + \int_\Omega AD^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon\right)\\ &= \int_\Omega f u = \langle \Lambda, D^\sigma u\rangle + \int_\Omega AD^\sigma u \cdot D^\sigma u\end{aligned}$$

and then

$$\varliminf_{\varepsilon\to 0} \int_\Omega \widehat{k_\varepsilon} |D^\sigma u^\varepsilon|^2 \leq \langle \Lambda, D^\sigma u\rangle.$$

Using $\widehat{k_\varepsilon}(|D^\sigma u^\varepsilon|^2 - g^2) \geq 0$, we obtain

$$\langle \Lambda, D^\sigma u\rangle \geq \varliminf_{\varepsilon\to 0} \int_\Omega \widehat{k_\varepsilon} |D^\sigma u^\varepsilon|^2 \geq \lim_{\varepsilon\to 0} \int_\Omega \widehat{k_\varepsilon} g^2 = \langle \lambda, g^2\rangle \geq \langle \lambda, |D^\sigma u|^2\rangle.$$

We also have

$$
\begin{aligned}
0 \leq \varliminf_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma(u^\varepsilon - u)|^2 &= \varliminf_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 - 2\lim_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon D^\sigma u^\varepsilon\cdot D^\sigma u\\
&\quad + \lim_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma u|^2\\
&\leq \langle \Lambda, D^\sigma u\rangle - 2\langle \Lambda, D^\sigma u\rangle + \langle \lambda, |D^\sigma u|^2\rangle\\
&= -\langle \Lambda, D^\sigma u\rangle + \langle \lambda, |D^\sigma u|^2\rangle,
\end{aligned}
$$

and therefore we conclude

$$
\langle \Lambda, D^\sigma u\rangle = \langle \lambda, |D^\sigma u|^2\rangle \quad\text{and}\quad \varliminf_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma(u^\varepsilon - u)|^2 = 0.
$$

Given $v \in \mathbb{K}_g$, we have

$$
\begin{aligned}
&\varliminf_{\varepsilon\to 0}\left|\int_\Omega \widehat{k}_\varepsilon D^\sigma(u^\varepsilon - u)\cdot D^\sigma v\right|\\
&\quad\leq \varliminf_{\varepsilon\to 0}\left(\int_\Omega \widehat{k}_\varepsilon |D^\sigma(u^\varepsilon - u)|^2\right)^{\frac12} \|\widehat{k}_\varepsilon\|^{\frac12}_{L^1(\Omega)}\|D^\sigma v\|_{L^\infty(\Omega)} = 0,
\end{aligned}
\tag{3.8}
$$

because, by estimate (3.2b), $\widehat{k}_\varepsilon$ is uniformly bounded in $L^1(\Omega)$. So, for any $v \in \mathbb{K}_g$,

$$
\begin{aligned}
\int_\Omega f v &= \varliminf_{\varepsilon\to 0}\int_\Omega (\widehat{k}_\varepsilon + A)D^\sigma u^\varepsilon\cdot D^\sigma v = \varliminf_{\varepsilon\to 0}\Big(\int_\Omega (\widehat{k}_\varepsilon + A)D^\sigma(u^\varepsilon - u)\cdot D^\sigma v\\
&\quad + \lim_{\varepsilon\to 0}\int_\Omega (\widehat{k}_\varepsilon + A)D^\sigma u\cdot D^\sigma v\Big) = \langle \lambda D^\sigma u, D^\sigma v\rangle + \int_\Omega A D^\sigma u\cdot D^\sigma v,
\end{aligned}
$$

concluding the proof of (1.8a).

Since $\displaystyle\int_\Omega \widehat{k}_\varepsilon v \geq 0$ for all $v \in L^\infty(\Omega)$ such that $v \geq 0$ then, for such v, we also have $\langle \lambda, v\rangle \geq 0$, which means that $\lambda \geq 0$.

For $v \in L^\infty(\Omega)$ set $v^+ = \max\{v, 0\}$, $v^- = (-v)^+$. Since $\widehat{k}_\varepsilon(|D^\sigma u^\varepsilon|^2 - g^2) \geq 0$ then

$$
\begin{aligned}
\langle \lambda, g^2 v^\pm\rangle &\leq \varliminf_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 v^\pm\\
&= \varliminf_{\varepsilon\to 0}\Big(\int_\Omega \widehat{k}_\varepsilon |D^\sigma(u^\varepsilon - u)|^2 v^\pm - 2\int_\Omega \widehat{k}_\varepsilon D^\sigma(u^\varepsilon - u)\cdot D^\sigma u v^\pm\\
&\quad + \int_\Omega \widehat{k}_\varepsilon |D^\sigma u|^2 v^\pm\Big)\\
&= \langle \lambda, |D^\sigma u|^2 v^\pm\rangle, \quad\text{using (3.8)},
\end{aligned}
$$

concluding that

$$\langle \lambda, (|D^\sigma u|^2 - g^2)\, v^\pm \rangle \geq 0.$$

The fact that $\widehat{k_\varepsilon} \geq 0$ and $u \in \mathbb{K}^\sigma_g$ imply $\widehat{k_\varepsilon}(|D^\sigma u|^2 - g^2)v^\pm \leq 0$ and, therefore, integrating and letting $\varepsilon \to 0$, $\langle \lambda, (D^\sigma u|^2 - g^2)\, v^\pm \rangle \leq 0$, and so

$$\langle \lambda, (|D^\sigma u|^2 - g^2)\, v \rangle = 0.$$

Writting $v = \frac{w}{|D^\sigma u| + g}$, for any $w \in L^\infty(\Omega)$, we conclude (1.8b). □

4 The Quasi-Variational Inequality with σ-Gradient Constraint

In this section we consider a map G such that

$$G : L^{2^*}(\Omega) \to L^\infty_\nu(\Omega) \tag{4.1}$$

is a continuous and bounded operator, where 2^* is the Sobolev exponent as in (2.5) for $0 < \sigma < 1$.

We set

$$\mathbb{K}^\sigma_{G[u]} = \left\{ v \in H^\sigma_0(\Omega) : |D^\sigma v| \leq G[u] \text{ a.e. in } \Omega \right\} \tag{4.2}$$

and we shall consider the quasi-variational inequality

$$u \in \mathbb{K}^\sigma_{G[u]} : \qquad \int_\Omega A D^\sigma u \cdot D^\sigma (v - u) \geq \int_\Omega f(v - u), \qquad \forall v \in \mathbb{K}^\sigma_{G[u]}. \tag{4.3}$$

Generalising a compactness argument of [6] where quasi-variational inequalities of this type were considered for the gradient case $\sigma = 1$, we may give a general existence theorem.

Theorem 4.1 *Under the assumptions* (2.1)*, for continuous and bounded operators G satisfying* (4.1) *and for any $f \in L^{2^\#}(\Omega)$, with $2^\#$ as in* (2.5)*, there exists at least one solution for the quasi-variational inequality* (4.3).

Proof Let $u = S(f, g)$ be the unique solution of the variational inequality (1.7) with $g = G[w]$ for any $w \in L^{2^*}(\Omega)$. If $C_* > 0$ denotes the Sobolev constant as in Theorem 2.1, since $f_2 = 0$ corresponds always to the solution $u_2 = 0$, we have the a priori estimate

$$\|u\|_{L^{2^*}(\Omega)} \leq C_* \|u\|_{H^\sigma_0(\Omega)} \leq \tfrac{C_*}{a_*} \|f\|_{L^{2^\#}(\Omega)} \equiv c_f, \tag{4.4}$$

independently of $g \in L^\infty_\nu(\Omega)$.

Set $B_{c_f} = \left\{ v \in L^{2^*}(\Omega) : \|v\|_{L^{2^*}(\Omega)} \leq c_f \right\}$ and define the nonlinear map $T = S \circ G : L^{2^*}(\Omega) \ni w \mapsto u \in L^{2^*}(\Omega)$ where $u = S(f, G[w]) \in \mathbb{K}^\sigma_{G[w]} \cap \mathscr{C}^{0,\beta}(\overline{\Omega})$, $0 < \beta < \sigma$ by (2.16).

Clearly, (4.4) implies $T(B_{c_f}) \subset B_{c_f}$ and, by the continuity of G and Theorem 2.2, T is also a continuous map. On the other hand, G is bounded, i.e. transforms bounded sets in $L^{2^*}(\Omega)$ into bounded sets of $L^\infty_\nu(\Omega)$ and $S \circ T$ is also a bounded operator. Therefore, by (2.16), $T(B_{c_f})$ is also a bounded set of $C^{0,\beta}(\overline{\Omega})$. Since the embedding $C^{0,\beta}(\overline{\Omega}) \hookrightarrow L^{2^*}(\Omega)$ is compact, the Schauder fixed point theorem guarantees the existence of $u = Tu$, which solves (4.3). □

Example 4.1 Consider the operator $G : L^{2^*}(\Omega) \to L^\infty_\nu(\Omega)$ defined as follows:

$$G[u](x) = F(x, w(x)), \tag{4.5}$$

where $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function bounded in $x \in \Omega$ and continuous in $w \in \mathbb{R}$, uniformly in $x \in \Omega$, satisfying, for some $\nu > 0$,

$$0 < \nu \leq F(x, w) \leq \varphi(|w|) \qquad \text{a.e. } x \in \Omega, \tag{4.6}$$

and for some monotone increasing function φ. We may choose

$$w(x) = \int_\Omega \vartheta(x, y) u(y)\, dy, \tag{4.7}$$

where we give $\vartheta \in L^\infty\big(\Omega_x; L^{2^\#}(\Omega_y)\big)$. For $u_n \underset{n}{\to} u$ in $L^{2^*}(\Omega)$, from the estimate

$$\begin{aligned}\sup_{x\in\Omega} |w_n(x) - w(x)| &= \sup_{x\in\Omega} \left| \int_\Omega \vartheta(x, y)(u_n(y) - u(y))dy \right| \\ &\leq \sup_{x\in\Omega} \|\vartheta(x, \cdot)\|_{L^{2\#}(\Omega)} \|u_n - u\|_{L^{2^*}(\Omega)}\end{aligned}$$

and by the uniform continuity of F, we have

$$\|G[u_n] - G[u]\|_{L^\infty(\Omega)} = \|F(w_n) - F(w)\|_{L^\infty(\Omega)} \underset{n}{\to} 0,$$

implying the continuity of G.

The boundedness of G is a consequence of (4.6) and therefore G satisfies the assumptions of Theorem 4.1.

Example 4.2 Consider now the operator $G : H^\sigma_0(\Omega) \to L^\infty_\nu(\Omega)$ given also by (4.5) with F under the same assumptions as in the previous example, but now with

$$w(x) = \Phi(u)(x) = \int_\Omega \Theta(x, y) \cdot D^\sigma u(y) dy, \tag{4.8}$$

where $\Theta \in \mathscr{C}\big(\overline{\Omega}_x; L^2(\Omega_y)^N\big)$. Now G is not only bounded but also completely continuous, since $\Phi : H^\sigma_0(\Omega) \to \mathscr{C}^0(\overline{\Omega})$ is also completely continuous. Indeed, if $u_n \underset{n}{\rightharpoonup} u$ in $H^\sigma_0(\Omega)$-weak, then $w_n = \Phi(u_n) \underset{n}{\to} \Phi(u) = w$ in $\mathscr{C}(\overline{\Omega})$, because $\{D^\sigma u_n\}_n$, being bounded in $L^2(\Omega)^N$ implies $\{w_n\}_n$ uniformly bounded in $\mathscr{C}^0(\overline{\Omega})$,

$$|w_n(x)| \le \|\Theta(x,\cdot)\|_{L^2(\Omega)^N} \|D^\sigma u_n\|_{L^2(\Omega)^N}, \qquad \forall x \in \overline{\Omega}$$

and also equicontinuous in $\overline{\Omega}$ by

$$|w_n(x) - w_n(z)| \le C\|\Theta(x,\cdot) - \Theta(z,\cdot)\|_{L^2(\Omega)^N}.$$

But G is not defined in the whole $L^{2^*}(\Omega)$ and therefore we cannot apply Theorem 4.1 to solve (4.3). Nevertheless, the solvability of (4.3) in this example is an immediate consequence of the following theorem.

Theorem 4.2 *Assume* (2.1) *and let* $f \in L^{2^\#}(\Omega)$ *as previously. If the nonlinear and nonlocal operator* G *satisfies*

$$G : H_0^\sigma(\Omega) \to L_\nu^\infty(\Omega) \ \text{ is bounded and completely continuous} \tag{4.9}$$

then there exists a solution u *to the quasi-variational inequality* (4.3).

Proof Due to the estimate (4.4) and the assumption (4.9), the proof is analogous by applying the Schauder fixed point theorem to the nonlinear completely continuous map

$$T = S \circ G : H_0^\sigma(\Omega) \ni w \mapsto u = S(f, G[w]) \in H_0^\sigma(\Omega).$$

□

Example 4.3 By restricting the domain of G and using the same type of Carathéodory function F as in Example 4.1, we can introduce the superposition operator

$$G[u](x) = F(x, u(x)), \qquad u \in \mathscr{C}^0(\overline{\Omega}),\ x \in \Omega. \tag{4.10}$$

In order to guarantee that $G : \mathscr{C}(\overline{\Omega}) \to L_\nu^\infty(\Omega)$ is a continuous and bounded operator in an appropriate space to obtain a fixed point, we need to require that the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded function in $x \in \Omega$ in each compact for the variable u, continuous in $u \in \mathbb{R}$ uniformly in $x \in \Omega$, and satisfying (4.6), where the monotone increasing function φ satisfies

$$0 < \nu \le \varphi(t) \le C_0 + C_1 t^{2^*/p}, \qquad t \in \mathbb{R}, \tag{4.11}$$

for some $p > \frac{N}{\sigma}$ and 2^* the Sobolev exponent as in (2.5).

This situation is covered by the next theorem, since the assumption (4.11) implies the condition (4.13) below.

Theorem 4.3 *Assume* (2.1)*, let* $f \in L^{2^\#}(\Omega)$ *and the functional* G *be such that*

$$G : \mathscr{C}^0(\overline{\Omega}) \to L_\nu^\infty(\Omega) \qquad \text{is a continuous operator} \tag{4.12}$$

and satisfying, for some positive monotone increasing function η,

$$\|G[w]\|_{L^p(\Omega)} \le \eta\big(\|w\|_{L^{2^*}(\Omega)}\big) \tag{4.13}$$

for some $p > \frac{N}{\sigma}$ *and* 2^* *the Sobolev exponent of* $H_0^\sigma(\Omega) \hookrightarrow L^{2^*}(\Omega)$. *Then there exists a solution of the quasi-variational inequality* (4.3).

Proof As before, we set $T = S \circ G : \mathscr{C}^0(\overline{\Omega}) \to H_0^\sigma(\Omega)$, where $u = S(f, G[w])$, for $w \in \mathscr{C}^0(\overline{\Omega})$ solves (1.7) with $g = G[w]$.

In order to apply the Leray-Schauder principle, we set

$$\mathscr{S} = \big\{w \in \mathscr{C}^0(\overline{\Omega}) : w = \theta T w,\ \theta \in [0,1]\big\}$$

and we show that $\mathscr{S}$ is a priori bounded. For any $w \in \mathscr{S}$, $u = Tw$ solves (1.7) with $g = G[w]$. Hence, by (2.4c) and the assumption (4.13) we have, noting that $w = \theta u$,

$$\begin{aligned}\|w\|_{\mathscr{C}^0(\overline{\Omega})} &\le C_\sigma \|D^\sigma w\|_{L^p(\Omega)^N} \le C_\sigma \theta \|G[w]\|_{L^p(\Omega)^N}\\ &\le C_\sigma \eta\big(\|w\|_{L^{2^*}(\Omega)}\big) \le C_\sigma \eta(c_f),\end{aligned}$$

by the a priori estimate (4.4).

Since, by (2.3), $T(\mathscr{C}^0(\overline{\Omega})) \hookrightarrow \mathscr{C}^{0,\beta}(\overline{\Omega}) \hookrightarrow \mathscr{C}^0(\overline{\Omega})$ and this last embedding is compact, we may conclude that T is a completely continuous mapping into a closed ball of $\mathscr{C}^0(\overline{\Omega})$ and its fixed point $u = Tu$ solves (4.3). □

It is clear that in general we cannot expect the uniqueness of solution to quasi-variational inequalities of the type (4.3). However, the Lipschitz continuity of the solution map $f \mapsto u$ to the variational inequality (1.7), given by Theorem 2.1, allows us to obtain, via the strict contraction Banach fixed point principle, a uniqueness result in a special case of "small" and controlled variations of the convex sets for the quasi-variational situation with separation of variables in the nonlocal constraint G.

We denote, for $R > 0$,

$$B_R = \big\{v \in H_0^\sigma(\Omega) : \|v\|_{H_0^\sigma(\Omega)} \le R\big\}.$$

Theorem 4.4 *Let* $f \in L^{2^\#}(\Omega)$, $\varphi \in L^\infty_\nu(\Omega)$ *and*

$$G[u](x) = \varphi(x)\Gamma(u), \qquad x \in \Omega, \tag{4.14}$$

where $\Gamma : H_0^\sigma(\Omega) \to \mathbb{R}^+$ *is a functional satisfying*

(i) $0 < \eta(R) \le \Gamma(u) \le E(R), \quad \forall u \in B_R$,
(ii) $|\Gamma(u_1) - \Gamma(u_2)| \le \gamma(R)\|u_1 - u_2\|_{H_0^\sigma(\Omega)}, \quad \forall u_1, u_2 \in B_R$,

for sufficiently large $R \in \mathbb{R}^+$, *with* η, E *and* γ *being monotone increasing positive functions of* R.

Then the quasi-variational inequality (4.3) *has a unique solution, provided*

$$2C_{\#}\,\frac{\gamma(R_f)}{\eta(R_f)}\,\|f\|_{L^{2^\#}(\Omega)} < 1, \tag{4.15}$$

where $R_f \equiv C_{\#}\|f\|_{L^{2^\#}(\Omega)}$ *with* $C_{\#} = C_*/a_*$ *and* C_* *is the constant of the Sobolev embedding as in* (4.4).

Proof Let $S : B_R \ni v \mapsto u \in H^\sigma_0(\Omega)$ be the solution map with $u = S(f, G[v])$ being the unique solution of the variational inequality (1.7) with $g = G[v]$.

The a priori estimate (4.4) implies $S(B_{R_f}) \subset B_{R_f}$.

Given $v_i \in B_R$, let $u_i = S(v_i) = S(f, \varphi\,\Gamma(v_i))$, $i = 1, 2$, and choose $\mu = \frac{\Gamma(v_2)}{\Gamma(v_1)} > 1$, without loss of generality.

Setting $g = \varphi\,\Gamma(v_1)$, we have $\mu\, g = \varphi\,\Gamma(v_2)$ and

$$\begin{aligned} S(\mu\, f, \mu\, g) &= \mu S(f, g),\\ \mu - 1 &= \frac{\Gamma(v_2) - \Gamma(v_1)}{\Gamma(v_1)} \le \frac{\gamma(R_f)}{\eta(R_f)}\|v_1 - v_2\|_\sigma \end{aligned}$$

by recalling the assumptions (i) and (ii) and denoting $\|w\|_\sigma = \|w\|_{H^\sigma_0(\Omega)}$ for simplicity.

Consequently, using (4.4) and (2.18) with $f_1 = f$ and $f_2 = \mu\, f$, we have

$$\begin{aligned} \|S(v_1) - S(v_2)\|_\sigma &\le \|S(f, g) - S(\mu f, \mu g)\|_\sigma + \|S(\mu f, \mu g) - S(f, \mu g)\|_\sigma\\ &\le (\mu - 1)\|u_1\|_\sigma + (\mu - 1)C_{\#}\|f\|_{L^{2^\#}(\Omega)}\\ &\le 2C_{\#}(\mu - 1)\|f\|_{L^{2^\#}(\Omega)}\\ &\le 2C_{\#}\frac{\gamma(R_f)}{\eta(R_f)}\|v_1 - v_2\|_\sigma\|f\|_{L^{2^\#}(\Omega)} \end{aligned}$$

and the conclusion of the theorem follows immediately. □

Example 4.4 We can take Γ of the form

$$\Gamma(u) = \int_\Omega e(y, u(y), D^\sigma u(y))\,dy, \quad u \in H^\sigma_0(\Omega),$$

with $e : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [\eta, \infty)$, for some $\eta > 0$, under a local Lipschitz condition of the type

$$|e(y, v, \xi) - e(y, w, \zeta)| \le \gamma(R)\big(|v - w| + |\xi - \eta|\big)$$

for $|v|$, $|w|$, $|\xi|$ and $|\zeta|$ less or equal to R.

Remark 4.1 Assumptions (i) and (ii) have been used in Appendiz B of [5] under the implicit assumptions of smallness of the term f, and in [12] in a simplified and more precise form in the case of gradient type (i.e. $\sigma = 1$) and for a class of general operators of p-Laplacian type.

Remark 4.2 The existence of solution of the quasi-variational inequality (4.3) is obtained in this section by finding a fixed point of the map $w \mapsto S(f, G[w]) = u$, under suitable assumptions. But when $u = S(f, G[w])$ is the solution of (1.7) then there exists $\lambda \in L^\infty(\Omega)'$ such that (u, λ) solves problem (1.8a)–(1.8b) with data $(f, G[w])$. In particular, when u is a fixed point $u = S(f, G[u])$ it solves the quasi-variational inequality, and we immediately get existence of a solution (λ, u) of problem (1.8a)–(1.8b) for the quasi-variational case.

Acknowledgements The authors would like to thank to the Reviewers for their helpful comments. The research of J. F. Rodrigues was partially done under the framework of the Project PTDC/MAT-PUR/28686/2017 at CMAFcIO/ULisboa and L. Santos was partially supported by the Centre of Mathematics the University of Minho through the Strategic Project PEst UID/MAT/0003/2013.

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