

## Research Article

Fernando Miranda, José Francisco Rodrigues\* and Lisa Santos

# Evolutionary quasi-variational and variational inequalities with constraints on the derivatives

<https://doi.org/10.1515/anona-2018-0113>

Received May 15, 2018; revised July 17, 2018; accepted September 2, 2018

**Abstract:** This paper considers a general framework for the study of the existence of quasi-variational and variational solutions to a class of nonlinear evolution systems in convex sets of Banach spaces describing constraints on a linear combination of partial derivatives of the solutions. The quasi-linear operators are of monotone type, but are not required to be coercive for the existence of weak solutions, which is obtained by a double penalization/regularization for the approximation of the solutions. In the case of time-dependent convex sets that are independent of the solution, we show also the uniqueness and the continuous dependence of the strong solutions of the variational inequalities, extending previous results to a more general framework.

**Keywords:** Evolutionary quasi-variational inequalities, gradient constraints, constraints on the derivatives

**MSC 2010:** 35R35, 35K87, 35K92, 47J20, 49J40

## 1 Introduction

While variational inequalities were introduced in 1964, by Fichera and Stampacchia in the framework of minimization problems with obstacle constraints, the first evolutionary variational inequality was solved in the seminal paper of Lions and Stampacchia [24], which was followed by many other works, including the extension to pseudo-monotone operators by Brézis in 1968 [7] (see also [23, 33]). Quasi-variational inequalities were introduced later by Bensoussan and Lions in 1973 to describe impulse control problems [5] and were developed for several other mathematical models with free boundaries (see, for instance, [3, 25]), mainly as implicit unilateral problems of obstacle type, in which the constraints depend on the solution.

The first physical models with gradient constraints formulated with quasi-variational inequalities of evolution type were proposed by Prigozhin, in [29] and [28], respectively for the sandpile growth and for the magnetization of type-II superconductors. This last model has motivated a first existence result for stationary problems in [21], including other applications in elastoplasticity and in electrostatics, and, in [31], in the parabolic framework for the  $p$ -Laplacian with an implicit gradient constraint, which was later extended to quasi-variational solutions for first-order quasilinear equations in [32], always in the scalar cases.

---

**Fernando Miranda, Lisa Santos**, CMAT – Departamento de Matemática, Escola de Ciências, Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal, e-mail: [fmiranda@math.uminho.pt](mailto:fmiranda@math.uminho.pt), [lisa@math.uminho.pt](mailto:lisa@math.uminho.pt)  
<https://orcid.org/0000-0002-7624-4816>, <https://orcid.org/0000-0003-0286-1616>

\***Corresponding author: José Francisco Rodrigues**, CMAFCIO – Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, P-1749-016 Lisboa, Portugal, e-mail: [jfrodriques@ciencias.ulisboa.pt](mailto:jfrodriques@ciencias.ulisboa.pt)  
<https://orcid.org/0000-0001-8438-0749>

In this work, we consider weak solutions  $\mathbf{u} = \mathbf{u}(x, t)$  to a class of quasi-variational inequalities associated with evolution equations or systems of the type

$$\partial_t \mathbf{u} + L^* \mathbf{a}(L\mathbf{u}) + \mathbf{b}(\mathbf{u}) = \mathbf{f} \quad (1.1)$$

formally in the unsaturated region of the scalar constraint

$$|L\mathbf{u}| \leq G[\mathbf{u}],$$

i.e. in the domain  $\{(x, t) : |L\mathbf{u}(x, t)| < G[\mathbf{u}](x, t)\}$ , with a nonlocal positive and compact operator  $G$ , where  $\partial_t \mathbf{u}$  denotes the partial time derivative,  $L$  is a linear partial differential operator in  $x$  with bounded coefficients and  $L^*$  is its formal dual. Here the monotone vector fields  $\mathbf{a}$  and  $\mathbf{b}$  are of power-type growth, and the boundary value problems may be coercive or not. However, in the region  $\{(x, t) : |L\mathbf{u}(x, t)| = G[\mathbf{u}](x, t)\}$  equation (1.1), in general, does not hold unless an extra term is added, raising interesting open questions. The general form of  $L$  covers, in particular, the gradient, the Laplacian and higher-order operators, the curl, the symmetric part of the Jacobian or classes of smooth vector fields such as those of Hörmander type. Weak quasi-variational solutions, which in general are non-unique and do not have the time derivative in the dual space of the solution, are obtained by the passages to the limit of two vanishing parameters, one for an appropriate approximation/penalization of the constraint on  $L\mathbf{u}$  and a second one for a coercive regularization, as in [32]. This method allows the application of the Schauder fixed point theorem to a general regularized two parameters variational equation of type (1.1) and extends considerably the work [2].

When the constraint  $G$ , which may depend on time and space, is independent of the solution, i.e.  $G = g(x, t)$ , the problem becomes a variational one with the solution belonging to a time dependent convex set of a suitable Banach space. In this case, if the vector fields  $\mathbf{a}$  and  $\mathbf{b}$  are monotone, there holds uniqueness of the weak solution. Under additional assumptions on the data, we show the existence, uniqueness and continuous dependence of the stronger solution of the corresponding evolution variational inequality, when the time derivative is actually an  $L^2$  function. Here our method is adapted to gradient-type constraints and it develops and extends the pioneer work of [34], which was continued in [35], extended to a  $p$ -curl system in [27] and to thick flows by [30] (see also [26]). Although variational inequalities with time dependent convex sets have been studied in several works (see, for instance, [17, 20] and the references therein), for the case of a convex with gradient constraint only a few results have been stated, namely in [6], as an application of abstract theorems, which assumptions are difficult to verify and, in general, require stronger hypotheses.

Recently, other approaches to evolutionary quasi-variational problems with gradient constraint have been developed by Kenmochi and co-workers in [11, 16, 18, 19], using variational evolution inclusions in Hilbert spaces with subdifferentials with a nonlocal dependence on parameters, and by Hintermüller and Rautenberg in [13], using the pseudo-monotonicity and the  $C_0$ -semigroup approach of Brézis-Lions, and in [14], using contractive iteration arguments that yield uniqueness results and numerical approximation schemes in interesting but special situations. Although the elegant and abstract approach of [13] yields the existence of weak quasi-variational solutions under general stability conditions of Mosco type and a general scheme for the numerical approximation of a solution, the required assumptions for the existence theory are somehow more restrictive than ours, in particular in what concerns the required strong coercive condition. Other recent results on evolutionary quasi-variational inequalities can be found in [20, 36], both in more abstract frameworks and oriented to unilateral-type problems and, therefore, with limited interest to constraints on the derivatives of the solutions. Recently, in [15], a semidiscretization in time, with monotone non-decreasing data, was used to obtain non-decreasing in time solutions to quasi-variational inequalities with gradient constraints, including an interesting numerical scheme.

This paper is organized as follows: In Section 2, we state our framework and the main results on the existence of weak quasi-variational solutions and on the well-posedness of the strong variational solutions. In Section 3, we illustrate the nonlocal constraint operator  $G$  and the linear partial differential operator  $L$  with several examples of applications. Section 4 deals with the approximated problem and a priori estimates. The proof of the existence of the weak quasi-variational solutions is given in Section 5 and, finally, in Section 6 we show the uniqueness and the continuous dependence on the data in the variational inequality case.

## 2 Assumptions and main results

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^d$  with a Lipschitz boundary,  $d \geq 2$  and for  $t \in (0, T]$  denote  $Q_t = \Omega \times (0, t)$ . For a real vector function  $\mathbf{u} = \mathbf{u}(x, t) = (u_1, \dots, u_m)$ ,  $(x, t) \in Q_T$ , and a multi-index  $\alpha = (\alpha_1, \dots, \alpha_d)$ , with  $\alpha_1, \dots, \alpha_d \in \mathbb{N}_0$  and  $\alpha_1 + \dots + \alpha_d = |\alpha|$ , we denote

$$\partial^\alpha u_i = \frac{\partial^{|\alpha|} u_i}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}},$$

the partial derivatives of  $u_i$ . Given real numbers  $a, b$ , we set  $a \vee b = \max\{a, b\}$  and  $a \wedge b = \min\{a, b\}$ .

We introduce now several assumptions which will be important to set the functional framework of our problem.

**Assumption 2.1.** For  $p \in [1, \infty]$ , let  $L$  be a linear differential operator of order  $s \geq 1$ , given by

$$L : \mathbf{V}_p \rightarrow L^p(\Omega)^\ell \quad \text{such that} \quad \mathbf{V}_p = \{\mathbf{u} \in L^p(\Omega)^m : L\mathbf{u} \in L^p(\Omega)^\ell\}$$

is endowed with the graph norm,  $\ell, m \in \mathbb{N}$ .

In general, the operator  $L$  can have the form

$$(L\mathbf{u})_j = \sum_{|\alpha| \leq s} \sum_{k=1}^m \lambda_{\alpha,k}^j \partial^\alpha u_k,$$

where  $j = 1, \dots, \ell$ ,  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$  is a multi-index and each  $\lambda_{\alpha,k}^j$  is in  $L^\infty(\Omega)$ , but we shall consider mainly the following four illustrative examples with constant coefficients, although we can consider also their generalizations with variable coefficients as in the fifth example:

- (i)  $Lu = \nabla u$  (gradient of  $u$ ,  $m = 1$ ,  $\ell = d$ ).
- (ii)  $Lu = \Delta u$  (Laplacian of  $u$ ,  $m = \ell = 1$ ).
- (iii)  $L\mathbf{u} = \nabla \times \mathbf{u}$  (curl of  $\mathbf{u}$ ,  $d = m = \ell = 3$ ).
- (iv)  $L\mathbf{u} = D\mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$  (symmetric part of the Jacobian of  $\mathbf{u}$ ,  $d = m$ ,  $\ell = m^2$ ).
- (v)  $L\mathbf{u} = (X_1 u, \dots, X_\ell u)$ , where  $X_j = \sum_{i=1}^d \alpha_{ji} \frac{\partial}{\partial x_i}$ , where  $\alpha_{ji}$  are appropriate scalar real functions (subelliptic gradient of  $u$ ,  $m = 1$ ,  $1 \leq j \leq \ell$ ,  $1 \leq i \leq d$ ).

**Assumption 2.2.** Let  $\mathbf{a} : Q_T \times \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$  and  $\mathbf{b} : Q_T \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  be Carathéodory functions, i.e. they are measurable functions in the variables  $(x, t)$  for all  $\boldsymbol{\xi} \in \mathbb{R}^\ell$  and  $\boldsymbol{\eta} \in \mathbb{R}^m$ , respectively, and they are continuous in the variables  $\boldsymbol{\xi} \in \mathbb{R}^\ell$  and  $\boldsymbol{\eta} \in \mathbb{R}^m$  for a.e.  $(x, t) \in Q_T$ . Suppose, additionally, that  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the following structural conditions: for all  $\boldsymbol{\xi}, \boldsymbol{\xi}' \in \mathbb{R}^\ell$  and  $\boldsymbol{\eta}, \boldsymbol{\eta}' \in \mathbb{R}^m$  and a.e.  $(x, t) \in Q_T$ ,

$$\begin{aligned} |\mathbf{a}(x, t, \boldsymbol{\xi})| &\leq a^* |\boldsymbol{\xi}|^{p-1}, \\ (\mathbf{a}(x, t, \boldsymbol{\xi}) - \mathbf{a}(x, t, \boldsymbol{\xi}')) \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}') &\geq 0, \\ |\mathbf{b}(x, t, \boldsymbol{\eta})| &\leq b^* |\boldsymbol{\eta}|^{(p-1) \vee 1}, \\ (\mathbf{b}(x, t, \boldsymbol{\eta}) - \mathbf{b}(x, t, \boldsymbol{\eta}')) \cdot (\boldsymbol{\eta} - \boldsymbol{\eta}') &\geq 0, \end{aligned} \tag{2.1}$$

where  $a^*$  and  $b^*$  are positive constants and  $1 < p < \infty$ .

**Assumption 2.3.** For a given  $p \in (1, \infty)$ , we work with a closed subspace  $\mathbb{X}_p$  of  $\mathbf{V}_p$  such that  $\mathbb{X}_p \subset L^2(\Omega)^m$  and  $\|\mathbf{v}\|_{\mathbb{X}_p} := \|L\mathbf{v}\|_{L^p(\Omega)^\ell}$  is a norm in  $\mathbb{X}_p$  equivalent to the norm induced from  $\mathbf{V}_p$ .

**Remark 2.4.** For simplicity, in this work we consider a functional framework where we suppose the Poincaré and Sobolev-type inequalities to be valid, as in the Dirichlet problems of the five examples. However, our approach is still valid for more general frameworks to include Neumann and mixed-type boundary conditions.

**Assumption 2.5.** There exists a Hilbert subspace  $\mathbb{H}$  of  $L^2(\Omega)^m$  such that  $(\mathbb{X}_p, \mathbb{H}, \mathbb{X}_p')$  is a Gelfand triple and the inclusion of  $\mathbb{X}_p$  into  $\mathbb{H}$  is compact for the given  $p$ ,  $1 < p < \infty$ .

From now on, we set

$$\mathcal{V}_p = L^p(0, T; \mathbb{X}_p), \quad \mathcal{H} = L^p(0, T; \mathbb{H}), \quad 1 \leq p \leq \infty,$$

and we observe that  $L^{p'}(0, T; \mathbb{X}'_p) = \mathcal{V}'_p$ , with  $p' = \frac{p}{p-1}$  for  $1 < p < \infty$ .

By well-known embedding theorems on Sobolev–Bochner spaces (see, for instance, [33, Chapter 7]), we have

$$\mathcal{B}_p = \{\mathbf{v} \in \mathcal{V}_p : \partial_t \mathbf{v} \in \mathcal{V}'_p\} \subset \mathcal{C}([0, T]; \mathbb{H}) \tag{2.2}$$

and Assumption 2.5 implies, by the Aubin–Lions lemma, that the embedding  $\mathcal{B}_p \hookrightarrow \mathcal{H}$  is also compact for  $1 < p < \infty$ .

**Assumption 2.6.** We consider a nonlinear continuous functional  $G : \mathcal{H} \rightarrow L^1(Q_T)$  such that its restriction to  $\mathcal{V}_p$  is compact with values in  $\mathcal{C}([0, T]; L^\infty(\Omega))$ , i.e.  $G : \mathcal{V}_p \rightarrow \mathcal{C}([0, T]; L^\infty(\Omega))$  is compact. In addition, we assume

$$0 < g_* \leq G[\mathbf{u}](x, t) \leq g^* \quad \text{for all } \mathbf{u} \in \mathcal{V}_p, \text{ for all } t \in [0, T] \text{ and a.e. } x \in \Omega,$$

for given constants  $g_*$  and  $g^*$ .

Since  $G$  is compact in  $\mathcal{V}_p$ , in particular for any sequence  $\{\mathbf{v}_n\}_n$  weakly convergent to  $\mathbf{v}$  in  $\mathcal{V}_p$ , there exists a subsequence, still denoted by  $\{\mathbf{v}_n\}_n$ , such that  $\{G[\mathbf{v}_n]\}_n$  converges uniformly to  $G[\mathbf{v}]$  in  $\mathcal{C}([0, T]; L^\infty(\Omega))$ .

For  $\mathbf{v} \in \mathcal{V}_p$  and a.e.  $t \in (0, T)$ , we define the nonempty convex set for  $G[\mathbf{v}](t) \in L^\infty(\Omega)$ :

$$\mathbb{K}_{G[\mathbf{v}](t)} = \{\mathbf{w} \in \mathbb{X}_p : |\mathbf{Lw}| \leq G[\mathbf{v}](t)\}, \tag{2.3}$$

where  $|\cdot|$  is the Euclidean norm in  $\mathbb{R}^\ell$  and we denote  $\mathbf{w} \in \mathbb{K}_{G[\mathbf{v}]}$  if and only if  $\mathbf{w}(t) \in \mathbb{K}_{G[\mathbf{v}](t)}$  for a.e.  $t \in (0, T)$ .

For  $1 \leq p < \infty$ , we denote the duality pairing between  $\mathbb{X}'_p$  and  $\mathbb{X}_p$  by  $\langle \cdot, \cdot \rangle_p$  and we consider the quasi-variational inequality associated with (1.1) and (2.3). Find  $\mathbf{u} \in \mathcal{V}_p$  satisfying

$$\left\{ \begin{array}{l} \mathbf{u} \in \mathbb{K}_{G[\mathbf{u}]}, \\ \int_0^T \langle \partial_t \mathbf{v}, \mathbf{v} - \mathbf{u} \rangle_p + \int_{Q_T} \mathbf{a}(\mathbf{Lu}) \cdot \mathbf{L}(\mathbf{v} - \mathbf{u}) + \int_{Q_T} \mathbf{b}(\mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \geq \int_{Q_T} \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) - \frac{1}{2} \int_{\Omega} |\mathbf{v}(0) - \mathbf{u}_0|^2, \\ \text{for all } \mathbf{v} \in \mathcal{B}_p \text{ such that } \mathbf{v} \in \mathbb{K}_{G[\mathbf{u}]} \end{array} \right. \tag{2.4}$$

**Theorem 2.7.** Suppose that Assumptions 2.1–2.6 are satisfied,  $\mathbf{f} \in L^2(Q_T)^m$  and  $\mathbf{u}_0 \in \mathbb{K}_{G[\mathbf{u}_0]}$ . Then the quasi-variational inequality (2.4) has a weak solution  $\mathbf{u} \in \mathcal{V}_p \cap L^\infty(0, T; L^2(\Omega)^m)$ .

We note that by Assumption 2.6 the solutions have bounded  $\mathbf{Lu}$  but, in general, this may not imply that  $\mathbf{u}$  is itself bounded. We also observe that, by insufficient regularity in time, we could not guarantee that the weak solution  $\mathbf{u}$  satisfies the initial condition in the classical sense, but only in the generalized sense (2.4) as in [7, 23].

We consider a positive bounded function  $g : Q_T \rightarrow \mathbb{R}^+$  and the special case of the convex set (2.3) with

$$G[\mathbf{v}](x, t) = g(x, t) \quad \text{for a.e. } t \in (0, T).$$

So,

$$\mathbf{v} \in \mathbb{K}_g \quad \text{if and only if} \quad \mathbf{v}(t) \in \mathbb{K}_{g(t)} = \{\mathbf{v} \in \mathbb{X}^p : |\mathbf{Lv}| \leq g(t)\} \text{ for a.e. } t \in (0, T). \tag{2.5}$$

In this case, the convex set being independent of the solution, the problem becomes variational and the weak solution of Theorem 2.7 is unique by the following theorem.

**Theorem 2.8.** The variational inequality (2.4) with a fixed convex  $\mathbb{K}_g$ , as in (2.5), for a given strictly positive function  $g = g(x, t) \in \mathcal{C}([0, T]; L^\infty(\Omega))$ ,  $\mathbf{u}_0 \in \mathbb{K}_{g(0)}$  and  $\mathbf{f} \in L^2(Q_T)^m$ , has at most one solution provided  $\mathbb{X}_p \subset \mathbb{H}$  and one of the monotonicity conditions is strict, i.e.

$$(\mathbf{b}(x, t, \boldsymbol{\eta}) - \mathbf{b}(x, t, \boldsymbol{\eta}')) \cdot (\boldsymbol{\eta} - \boldsymbol{\eta}') > 0 \quad \text{for } \boldsymbol{\eta} \neq \boldsymbol{\eta}',$$

or

$$(\mathbf{a}(x, t, \boldsymbol{\xi}) - \mathbf{a}(x, t, \boldsymbol{\xi}')) \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}') > 0 \quad \text{for } \boldsymbol{\xi} \neq \boldsymbol{\xi}',$$

and Assumption 2.3 holds.

We can now introduce the strong formulation of the corresponding variational inequality. Find

$$\mathbf{w} \in \mathcal{V}_p \cap H^1(0, T; L^2(\Omega)^m)$$

satisfying, for all  $t \in (0, T]$ ,

$$\begin{cases} \mathbf{w} \in \mathbb{K}_g, & \mathbf{w}(0) = \mathbf{w}_0, \\ \int_{Q_t} \partial_t \mathbf{w} \cdot (\mathbf{v} - \mathbf{w}) + \int_{Q_t} \mathbf{a}(L\mathbf{w}) \cdot L(\mathbf{v} - \mathbf{w}) + \int_{Q_t} \mathbf{b}(\mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \geq \int_{Q_t} \mathbf{f} \cdot (\mathbf{v} - \mathbf{w}), \\ \text{for all } \mathbf{v} \in \mathbb{K}_g \cap \mathcal{V}_p. \end{cases} \quad (2.6)$$

We observe that if  $\mathbf{w}$  is a strong solution to (2.6), it is also a weak solution to (2.4). Indeed, if we take  $\mathbf{v} \in \mathcal{B}_p \cap \mathbb{K}_g \subset \mathcal{C}(0, T; L^2(\Omega)^m)$  in (2.6) with  $t = T$ , since

$$\int_0^T \langle \partial_t \mathbf{w} - \partial_t \mathbf{v}, \mathbf{v} - \mathbf{w} \rangle_p = \frac{1}{2} \int_{\Omega} |\mathbf{v}(0) - \mathbf{w}_0|^2 - \frac{1}{2} \int_{\Omega} |\mathbf{v}(T) - \mathbf{w}(T)|^2 \leq \frac{1}{2} \int_{\Omega} |\mathbf{v}(0) - \mathbf{w}_0|^2,$$

we immediately conclude that  $\mathbf{w}$  also satisfies (2.4).

We consider also a stronger non-coercive framework with a potential vector field  $\mathbf{a}$  and a lower-order term  $\mathbf{b}$  with linear growth, by replacing Assumption 2.2 by the following.

**Assumption 2.9.** Let  $\mathbf{a} : Q_T \times \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$  and  $\mathbf{b} : Q_T \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  be Carathéodory functions, i.e. they are measurable functions in the variables  $(x, t)$  for all  $\boldsymbol{\xi} \in \mathbb{R}^\ell$  and  $\boldsymbol{\eta} \in \mathbb{R}^m$ , respectively, and they are continuous in the variables  $\boldsymbol{\xi} \in \mathbb{R}^\ell$  and  $\boldsymbol{\eta} \in \mathbb{R}^m$ , respectively, for a.e.  $(x, t) \in Q_T$ . Suppose, additionally, that there exists  $A : Q_T \times \mathbb{R}^\ell \rightarrow \mathbb{R}$  such that, for all  $\boldsymbol{\xi} \in \mathbb{R}^\ell$  and a.e.  $(x, t) \in \overline{Q_T}$ , the function  $A$  is differentiable in  $t$  and in  $\boldsymbol{\xi}$ , and

$$A = A(x, t, \boldsymbol{\xi}) \text{ is convex in } \boldsymbol{\xi}, \quad \nabla_{\boldsymbol{\xi}} A = \mathbf{a}, \quad (2.7)$$

$$0 \leq A(x, t, \boldsymbol{\xi}) \leq a^* |\boldsymbol{\xi}|^p, \quad |\partial_t A(x, t, \boldsymbol{\xi})| \leq A_1 + A_2 |\boldsymbol{\xi}|^p, \quad (2.8)$$

and  $\mathbf{b}$  satisfies the following structural conditions: for all  $\boldsymbol{\eta}, \boldsymbol{\eta}' \in \mathbb{R}^m$  and a.e.  $(x, t) \in Q_T$ ,

$$\begin{aligned} (\mathbf{b}(x, t, \boldsymbol{\eta}) - \mathbf{b}(x, t, \boldsymbol{\eta}')) \cdot (\boldsymbol{\eta} - \boldsymbol{\eta}') &\geq 0, \\ |\mathbf{b}(x, t, \boldsymbol{\eta})| &\leq b^* |\boldsymbol{\eta}|, \end{aligned} \quad (2.9)$$

where  $a^*$ ,  $A_1$ ,  $A_2$  and  $b^*$  are positive constants,  $1 < p < \infty$ .

In the non-coercive case, we have the well-posedness result on the existence, uniqueness and continuous dependence of the (strong) variational solution (2.6). Under the additional strong monotonicity assumption, for instance for operators of  $p$ -Laplacian type, when  $\mathbf{a}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2} \boldsymbol{\xi}$ , the continuous dependence result in the coercive case also holds in the space  $\mathcal{V}_p$ .

**Theorem 2.10.** Suppose that Assumptions 2.1, 2.3, 2.5 and 2.9 are satisfied and

$$\mathbf{f} \in L^2(Q_T)^m, \quad \mathbf{w}_0 \in \mathbb{K}_{g(0)}, \quad g \in W^{1,\infty}(0, T; L^\infty(\Omega)) \quad \text{with } g \geq g_* > 0. \quad (2.10)$$

Then the variational inequality (2.6) has a unique solution  $\mathbf{w} \in \mathcal{V}_p \cap H^1(0, T; L^2(\Omega)^m)$ .

**Theorem 2.11.** Suppose that the assumptions of Theorem 2.10 hold and for  $i = 1, 2$  let  $\mathbf{w}_i$  be the solution to the variational inequality (2.6) with data  $\mathbf{f}_i$ ,  $\mathbf{w}_{i_0}$ ,  $g_i$  satisfying (2.10). Then there exists a positive constant  $C = C(T)$  such that

$$\|\mathbf{w}_1 - \mathbf{w}_2\|_{L^\infty(0, T; L^2(\Omega)^m)}^2 \leq C(\|\mathbf{f}_1 - \mathbf{f}_2\|_{L^2(Q_T)^m}^2 + \|\mathbf{w}_{1_0} - \mathbf{w}_{2_0}\|_{L^2(\Omega)^m}^2 + \|g_1 - g_2\|_{L^1(0, T; L^\infty(\Omega))}). \quad (2.11)$$

If, in addition,  $\mathbf{a}$  satisfies

$$(\mathbf{a}(x, t, \boldsymbol{\xi}) - \mathbf{a}(x, t, \boldsymbol{\xi}')) \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}') \geq \begin{cases} a_* |\boldsymbol{\xi} - \boldsymbol{\xi}'|^p & \text{if } p \geq 2, \\ a_* (|\boldsymbol{\xi}| + |\boldsymbol{\xi}'|)^{p-2} |\boldsymbol{\xi} - \boldsymbol{\xi}'|^2 & \text{if } p < 2, \end{cases} \quad (2.12)$$

where  $a_*$  is a positive constant depending on  $p$ ,  $1 < p < \infty$ , then there exists  $C_* = C(a_*, p, T) > 0$  such that

$$\begin{aligned} & \|\mathbf{w}_1 - \mathbf{w}_2\|_{L^\infty(0, T; L^2(\Omega)^m)}^2 + \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathcal{Y}_p}^{2vp} \\ & \leq C_* (\|\mathbf{f}_1 - \mathbf{f}_2\|_{L^2(Q_T)^m}^2 + \|\mathbf{w}_{10} - \mathbf{w}_{20}\|_{L^2(\Omega)^m}^2 + \|\mathbf{g}_1 - \mathbf{g}_2\|_{L^1(0, T; L^\infty(\Omega))}). \end{aligned} \quad (2.13)$$

**Remark 2.12.** For strong solutions  $\mathbf{w} \in \mathbb{K}_g \cap H^1(0, T; L^2(\Omega)^m)$ , the variational inequality (2.6) is, for a.e.  $t \in (0, T)$ , equivalent to

$$\begin{aligned} & \int_{\Omega} \partial_t \mathbf{w}(t) \cdot (\mathbf{z} - \mathbf{w}(t)) + \int_{\Omega} \mathbf{a}(t, L\mathbf{w}(t)) \cdot L(\mathbf{z} - \mathbf{w}(t)) + \int_{\Omega} \mathbf{b}(t, \mathbf{w}(t)) \cdot (\mathbf{z} - \mathbf{w}(t)) \\ & \geq \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{z} - \mathbf{w}(t)) \quad \text{for all } \mathbf{z} \in \mathbb{K}_{g(t)}, \end{aligned} \quad (2.14)$$

provided we assume  $\mathbf{g} \in \mathcal{C}([0, T]; L^\infty(\Omega))$ ,  $\mathbf{g} \geq \mathbf{g}_* > 0$ . Indeed, for arbitrary  $\delta > 0$ ,  $0 < \delta < t < T - \delta$ , for fixed  $t \in (0, T)$ , we set

$$\varepsilon_\delta = \sup_{t-\delta < \tau < t+\delta} \|\mathbf{g}(t) - \mathbf{g}(\tau)\|_{L^\infty(\Omega)}$$

and we may define

$$\mathbf{v}(\tau) = \begin{cases} \mathbf{0} & \text{if } \tau \notin (t - \delta, t + \delta), \\ \frac{\mathbf{g}_*}{\mathbf{g}_* + \varepsilon_\delta} \mathbf{z} & \text{if } \tau \in (t - \delta, t + \delta), \end{cases}$$

which is such that  $\mathbf{v} \in \mathcal{Y}_p \cap \mathbb{K}_g$  whenever  $\mathbf{z} \in \mathbb{K}_{g(t)}$ . Hence we can choose this  $\mathbf{v}$  as test function in (2.6) with  $t = T$ , divide by  $2\delta$  and let  $\delta \rightarrow 0$  obtaining, by Lebesgue's theorem, inequality (2.14) for a.e.  $t \in (0, T)$ .

As a Corollary of Theorems 2.8 and 2.10, we can drop the differentiability in time of  $\mathbf{g}$  and still obtain an existence and uniqueness result for the weak variational inequality (2.4) with  $\mathbb{K}_g$ , extending [18, Theorem 3.8].

**Theorem 2.13.** Suppose that Assumptions 2.1, 2.3, 2.5 and 2.9 are satisfied and

$$\mathbf{f} \in L^2(Q_T)^m, \quad \mathbf{g} \in \mathcal{C}([0, T]; L^\infty(\Omega)) \quad \text{with } \mathbf{g} \geq \mathbf{g}_* > 0, \quad \mathbf{w}_0 \in \mathbb{K}_{g(0)}.$$

Then the variational inequality (2.4) for  $\mathbb{K}_g$  has a unique weak solution  $\mathbf{w} \in \mathcal{Y}_p \cap \mathcal{C}([0, T]; L^2(\Omega)^m)$ .

### 3 Applications with particular $G$ and $L$

In this section, we present some examples of compact nonlocal operators  $G$  satisfying Assumption 2.6, and linear operators  $L$  satisfying Assumption 2.1.

#### 3.1 Nonlocal compact operators

Here we are interested in two examples of compact operators  $G$  given in the form

$$G[\mathbf{v}] = g(x, t, \boldsymbol{\zeta}(\mathbf{v})(x, t)) \quad \text{a.e. in } Q_T, \quad (3.1)$$

where  $g = g(x, t, \boldsymbol{\zeta}) : Q_T \times \mathbb{R}^m \rightarrow \mathbb{R}$  is a positive function, continuous in  $(x, t) \in Q_T$  and in  $\boldsymbol{\zeta} \in \mathbb{R}^m$ , and  $\boldsymbol{\zeta} : \mathcal{Y}_p \rightarrow \mathcal{C}(\overline{Q}_T)^m$  is a completely continuous mapping.

### 3.1.1 Regularization by integration in time

We define the compact operator by

$$\zeta(\mathbf{v})(x, t) = \int_0^t \mathbf{v}(x, s)K(t, s) ds \quad \text{for a.e. } (x, t) \in Q_T,$$

where  $K = K(t, s)$  is a given kernel satisfying

$$K, \partial_t K \in L^\infty((0, T) \times (0, T)). \tag{3.2}$$

For simplicity, we assume here the existence of a constant  $g_*$  and a real bounded function  $g^*$  such that

$$0 < g_* \leq g(x, t, \xi) \leq g^*(M) \quad \text{for a.e. } (x, t) \in Q_T \text{ and for all } \xi : |\xi| \leq M.$$

We also assume that the embedding

$$\mathbb{X}_p \hookrightarrow \mathcal{C}(\overline{\Omega})^m \quad \text{is compact,} \tag{3.3}$$

which, by the Rellich–Kondratchov theorem, is satisfied if  $\mathbb{X}_p \subset W^{s,q}(\Omega)^m$  with  $q > \frac{d}{s}$ .

Suppose  $\mathcal{V}_p = L^p(0, T; \mathbb{X}_p)$ ,  $p > 1$ , and observe that, by assumption (3.2), not only  $\zeta(\mathbf{v}) \in \mathcal{V}_p$  but also  $\partial_t \zeta(\mathbf{v}) \in \mathcal{V}_p$ , i.e.

$$\zeta(\mathbf{v}) \in W^{1,p}(0, T; \mathbb{X}_p).$$

Hence, by [4, Lemma 2.2], for instance the image by  $\zeta$  of a bounded subset of  $\mathcal{V}_p$ , being bounded in  $W^{1,p}(0, T; \mathbb{X}_p)$ , by (3.3) is relatively compact in  $\mathcal{C}(\overline{Q}_T)^m$ . So  $\zeta : \mathcal{V}_p \rightarrow \mathcal{C}(\overline{Q}_T)^m$  is a completely continuous mapping, and therefore  $G$  defined in (3.1) satisfies Assumption 2.6.

### 3.1.2 Coupling with a nonlinear parabolic equation

We may define the compact operator through the unique solution of the Cauchy-Dirichlet problem for the quasilinear parabolic scalar equation

$$\partial_t \zeta - \nabla \cdot \mathbf{a}(x, t, \nabla \zeta) = \varphi_{\mathbf{v}} \quad \text{in } Q_T, \tag{3.4}$$

$$\zeta = 0 \quad \text{on } \partial\Omega \times (0, T), \quad \zeta(0) = \zeta_0 \quad \text{on } \Omega, \tag{3.5}$$

where  $\varphi_{\mathbf{v}} = \varphi(x, t)$  depends on  $\mathbf{v} \in \mathcal{V}_p$ , and the vector field  $\mathbf{a}$  satisfies (2.1) and (2.12) with  $p = 2$  and  $\ell = d$ .

It is well known that for each  $\varphi \in L^2(Q_T)$  and  $\zeta_0 \in L^2(\Omega)$ , the weak solution

$$\zeta \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

to (3.4), (3.5) and (2.6) depends continuously, in these spaces, on the variation of  $\varphi$  in the weak topology of  $L^2(Q_T)$ . Moreover, if  $\zeta_0 \in \mathcal{C}^\gamma(\overline{\Omega})$  is Hölder continuous for some  $0 < \gamma < 1$  and  $\varphi \in L^q(Q_T)$  for  $q > \frac{d+2}{2}$ , the following estimate holds (see [22, p. 419]):

$$\|\zeta\|_{\mathcal{C}^\lambda(\overline{Q}_T)} \leq C(\|\zeta_0\|_{\mathcal{C}^\gamma(\overline{\Omega})} + \|\varphi\|_{L^q(Q_T)})$$

for some  $\lambda$ ,  $0 < \lambda \leq \gamma < 1$ , where  $C > 0$  is a constant independent of the data  $\varphi$ .

Now, for each  $\mathbf{v} \in \mathcal{V}_p$  with  $p > \frac{d+2}{2}$  ( $p = 2$  if  $d = 1$ ) and given  $\boldsymbol{\psi} \in L^\infty(Q_T)^m$  and  $\boldsymbol{\eta} \in L^\infty(Q_T)^\ell$ , we may choose  $\zeta = \zeta(\mathbf{v})$  in (3.1) as being the solution of (3.4)–(3.5), with a given  $\zeta_0 \in \mathcal{C}^\gamma(\overline{\Omega})$  and

$$\varphi_{\mathbf{v}} = \varphi_0 + \boldsymbol{\psi} \cdot \mathbf{v} + \boldsymbol{\eta} \cdot L\mathbf{v} \in L^p(Q_T) \tag{3.6}$$

for some fixed  $\varphi_0 \in L^p(Q_T)$ . Hence, by (3.6) and the Ascoli theorem, the mapping  $\mathbf{v} \mapsto \varphi_{\mathbf{v}} \mapsto \zeta(\mathbf{v})$  is completely continuous from  $\mathcal{V}_p$  into  $\mathcal{C}(\overline{Q}_T)$ . Indeed, if  $\mathbf{v}_n \rightharpoonup \mathbf{v}$  in  $\mathcal{V}_p$ ,  $\{\zeta(\mathbf{v}_n)\}_n$  is bounded in

$$\mathcal{C}^\lambda(\overline{Q}_T) \cap L^2(0, T; H_0^1(\Omega))$$

and, for some subsequence,  $\zeta(\mathbf{v}_n) \rightharpoonup \zeta$  weakly in  $L^2(0, T; H_0^1(\Omega))$  and uniformly in  $\overline{Q_T}$ , for a

$$\zeta \in L^2(0, T; H_0^1(\Omega)) \cap \mathcal{C}(\overline{Q_T}),$$

where we have  $\zeta = \zeta(\mathbf{v})$  by monotonicity and uniqueness of the solution of (3.4)–(3.5). Then the whole sequence converges and the complete continuity of  $G = G[\mathbf{v}]$ , from  $\mathcal{V}_p$  into  $\mathcal{C}(\overline{Q_T})$ , is guaranteed by the assumptions.

We observe that, if  $\mathbb{X}_p \subset W^{s,p}(\Omega)^m$ ,  $s = 1, 2, \dots$ , in (3.4) we can also choose

$$\varphi_{\mathbf{v}} = \varphi_0 + \sum_{0 \leq |\alpha| \leq s} \boldsymbol{\psi}_{\alpha} \cdot \partial^{\alpha} \mathbf{v},$$

with  $\boldsymbol{\psi}_{\alpha} \in L^{\infty}(Q_T)^m$ , provided  $p > \frac{d+2}{2}$ , and even more general terms involving linear combinations of the gradients of the  $\boldsymbol{\psi}_{\alpha} \cdot \partial^{\alpha} \mathbf{v} \in L^p(Q_T)$ ,  $0 \leq |\alpha| \leq s$ , provided  $p > d + 2$ .

## 3.2 Linear differential operators

In this subsection, we illustrate some concrete results for the operators  $L$  referred to as examples in Section 1 for convex sets of the type (2.3). For simplicity, in all the examples we consider the vector fields

$$\mathbf{a}(\boldsymbol{\xi}) = \alpha |\boldsymbol{\xi}|^{p-2} \boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \mathbb{R}^{\ell}, \quad \alpha = \alpha(x, t) \geq 0 \text{ a.e. in } Q_T, \quad \mathbf{b} \equiv \mathbf{0},$$

and we assume that the operator  $G$  satisfies Assumption 2.6.

### 3.2.1 A problem with gradient constraint

**Corollary 3.1.** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^d$  with a Lipschitz boundary, let  $\mathcal{V}_p = L^p(0, T; W_0^{1,p}(\Omega))$  and let  $p > \max\{1, \frac{2d}{d+2}\}$ . Let further  $f \in L^2(Q_T)$  and  $u_0 \in \mathbb{K}_{G[u_0]}$ . Then the following quasi-variational inequality has a weak solution:*

$$\left\{ \begin{array}{l} u \in \mathbb{K}_{G[u]}, \\ \int_0^T \langle \partial_t v, v - u \rangle_p + \int_{Q_T} \alpha |\nabla u|^{p-2} \nabla u \cdot \nabla (v - u) \geq \int_{Q_T} f(v - u) - \frac{1}{2} \int_{\Omega} |v(0) - u_0|^2, \\ \text{for all } v \in \mathcal{V}_p \text{ such that } v \in \mathbb{K}_{G[u]}. \end{array} \right.$$

Actually, with  $Lu = \nabla u$  and  $V_p = W^{1,p}(\Omega)$ , Assumptions 2.1–2.6 are satisfied because the inclusion of  $\mathbb{X}_p = W_0^{1,p}(\Omega)$  into  $\mathbb{H} = L^2(\Omega)$  is compact for  $p > \max\{1, \frac{2d}{d+2}\}$ .

The degenerate case  $\alpha \equiv 0$  corresponds to the variational model of sandpile growth where  $G$  models the slope of the pile (see [29]). In [29], Prigozhin introduces an operator  $G$  which is discontinuous in the height  $u$  of the sandpile and leads to a quasi-variational formulation that is still an open problem.

### 3.2.2 A problem with Laplacian constraint

**Corollary 3.2.** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^d$  with a  $\mathcal{C}^{1,1}$  boundary, and let  $\mathcal{V}_p = L^p(0, T; W_0^{2,p}(\Omega))$  with  $p > \max\{1, \frac{2d}{d+4}\}$ . Let  $f \in L^2(Q_T)$  and  $u_0 \in \mathbb{K}_{G[u_0]}$ . Then the following quasi-variational inequality has a weak solution:*

$$\left\{ \begin{array}{l} u \in \mathbb{K}_{G[u]}, \\ \int_0^T \langle \partial_t v, v - u \rangle_p + \int_{Q_T} \alpha |\Delta u|^{p-2} \Delta u \Delta (v - u) \geq \int_{Q_T} f(v - u) - \frac{1}{2} \int_{\Omega} |v(0) - u_0|^2, \\ \text{for all } v \in \mathcal{V}_p \text{ such that } v \in \mathbb{K}_{G[u]}. \end{array} \right.$$

Here we choose

$$V_p = \{v \in L^p(\Omega) : \Delta v \in L^p(\Omega)\},$$

i.e. the operator  $L$  is the Laplacian. The subspace  $\mathbb{X}_p = W_0^{2,p}(\Omega)$  is endowed with the norm

$$\|v\|_{\mathbb{X}_p} = \|\Delta v\|_{L^p(\Omega)},$$

which is equivalent to the usual norm of  $W^{2,p}(\Omega)$  because  $\Delta$  is an isomorphism between  $\mathbb{X}_p$  and  $L^p(\Omega)$ . Besides,  $(\mathbb{X}_p, L^2(\Omega), \mathbb{X}'_p)$  is a Gelfand triple and the inclusion  $\mathbb{X}_p \subset L^2(\Omega)$  is compact because  $p > \max\{1, \frac{2d}{d+4}\}$ .

### 3.2.3 A problem with curl constraint

**Corollary 3.3.** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^3$  with a Lipschitz boundary, and let  $p > \frac{6}{5}$  and  $f \in L^2(Q_T)$ . Define*

$$\mathbb{X}_p = \{v \in L^p(\Omega)^3 : \nabla \times v \in L^p(\Omega)^3, \nabla \cdot v = 0, v \cdot n|_{\partial\Omega} = 0\}$$

or

$$\mathbb{X}_p = \{v \in L^p(\Omega)^3 : \nabla \times v \in L^p(\Omega)^3, \nabla \cdot v = 0, v \times n|_{\partial\Omega} = 0\}.$$

If  $u_0 \in \mathbb{K}_{G[u_0]}$ , the following quasi-variational inequality has a weak solution:

$$\left\{ \begin{array}{l} u \in \mathbb{K}_{G[u]}, \\ \int_0^T \langle \partial_t v, v - u \rangle_p + \int_{Q_T} \alpha |\nabla \times u|^{p-2} \nabla \times v \cdot \nabla \times (v - u) \geq \int_{Q_T} f \cdot (v - u) - \frac{1}{2} \int_{\Omega} |v(0) - u_0|^2, \\ \text{for all } v \in \mathcal{V}_p \text{ such that } v \in \mathbb{K}_{G[u]}. \end{array} \right.$$

Here  $Lv = \nabla \times v$  and  $\mathcal{V}_p = \{v \in L^p(\Omega)^3 : \nabla \times v \in L^p(\Omega)^3\}$ . In both choices of  $\mathbb{X}_p$ , corresponding to different boundary conditions, it is well known that  $\mathbb{X}_p$  is a closed subspace of  $W^{1,p}(\Omega)^3$  and that the semi-norm  $\|\nabla \times \cdot\|_{L^p(\Omega)^3}$  is a norm equivalent to the one induced in  $\mathbb{X}_p$  by the usual norm in  $W^{1,p}(\Omega)^3$  (for details see [1]). Here  $\mathbb{X}_p$  is compactly embedded in  $\mathbb{H} = \{v \in L^2(\Omega)^3 : \nabla \cdot v = 0\}$ .

This model is related to the Bean-type superconductivity variational inequality, which was solved in [27], with prescribed critical threshold  $G$ . If we let here this threshold be, for instance, dependent on the temperature  $\zeta$  defined by (3.4) and (3.5) and we impose  $p > \frac{5}{2}$ , we obtain the existence of a weak solution to the corresponding thermal and electromagnetic coupled problem.

### 3.2.4 Non-Newtonian thick fluids – a problem with a constraint on $D$

Set

$$\begin{aligned} Du &= \frac{1}{2}(\nabla u + \nabla u^T), \\ V_p &= \{v \in L^p(\Omega)^d : Dv \in L^p(\Omega)^{d^2}\}, \quad \mathbb{J} = \{v \in \mathcal{D}(\Omega)^d : \nabla \cdot v = 0\}, \\ \mathbb{X}_p &= \overline{\mathbb{J}}^{W^{1,p}(\Omega)^d} \quad \text{for } p > 1, d \geq 2. \end{aligned}$$

Let  $\mathcal{V}_p = L^p(0, T; V_p)$  and observe that  $\mathbb{X}_p$  is compactly embedded in

$$\mathbb{H} = \{v \in L^2(\Omega)^d : \nabla \cdot v = 0\} = \overline{\mathbb{J}}^{L^2(\Omega)^d}$$

if  $p > \frac{2d}{d+2}$ , by the Sobolev and Korn inequalities. Hence, using the results of [26, 30] for the variational inequality for incompressible thick fluids in the simpler case of the Stokes flow, we obtain the following conclusion.

**Corollary 3.4.** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^d$  with a Lipschitz boundary, and let  $d \geq 2$ ,  $p > \frac{2d}{d+2}$ ,  $f \in L^2(Q_T)^d$  and  $\mathbf{u}_0 \in \mathbb{K}_{G[\mathbf{u}_0]}$ . Then the quasi-variational inequality*

$$\left\{ \begin{array}{l} \mathbf{u} \in \mathbb{K}_{G[\mathbf{u}]}, \\ \int_0^T \langle \partial_t \mathbf{v}, \mathbf{v} - \mathbf{u} \rangle_p + \int_{Q_T} \alpha |D\mathbf{u}|^{p-2} D\mathbf{u} \cdot D(\mathbf{v} - \mathbf{u}) \geq \int_{Q_T} \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) - \frac{1}{2} \int_{\Omega} |\mathbf{v}(0) - \mathbf{u}_0|^2, \\ \text{for all } \mathbf{v} \in \mathcal{V}_p \text{ such that } \mathbf{v} \in \mathbb{K}_{G[\mathbf{u}]} \end{array} \right.$$

has a weak solution.

### 3.2.5 A problem with first-order vector fields constraint

Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , be a connected bounded open set and let  $L = (X_1, \dots, X_\ell)$  be a family of Lipschitz vector fields on  $\mathbb{R}^d$  that connect the space. We shall assume that the regularity of  $\partial\Omega$  and the structure of  $L$  support the following Sobolev–Poincaré compact embedding for  $p \geq 2$ :

$$\mathbb{X}_p \hookrightarrow L^2(\Omega). \quad (3.7)$$

This is the case of an Hörmander operator with

$$X_j = \sum_{i=1}^d \alpha_{ij} \partial_{x_i}, \quad j = 1, \dots, \ell,$$

with  $\alpha_{ij} \in \mathcal{C}^\infty(\overline{\Omega})$  such that the Lie algebra generated by these  $\ell$  vector fields has dimension  $d$ , when the set  $\mathbb{X}_p$  is the closure of  $\mathcal{D}(\Omega)$  in

$$V_p = \{v \in L^p(\Omega) : X_j v \in L^p(\Omega), j = 1, \dots, \ell\} \quad \text{with } p \geq 2,$$

with the graph norm and  $\partial\Omega \in \mathcal{C}^\infty$ . Indeed, in this case, it is known (see [9, 10, 12]) that the extension of the Rellich–Kondratchov theorem,

$$\mathbb{X}_p = \overline{\mathcal{D}(\Omega)}^{V_p} \hookrightarrow L^2(\Omega) \quad \text{is compact for } p \geq 2,$$

holds, and so  $(\mathbb{X}_p, L^2(\Omega), \mathbb{X}'_p)$  is a Gelfand triple with compact embeddings. For other classes of vector fields, namely associated with degenerate subelliptic operators, and a characterization of domains where (3.7) holds, see, for instance, [8, 12]. By the application of Theorem 2.7 we can now conclude the following existence result.

**Corollary 3.5.** *Suppose that  $\Omega$  is a bounded open subset of  $\mathbb{R}^d$  with a smooth boundary. Under assumption (3.7), if  $p \geq 2$ ,  $f \in L^2(Q_T)$  and  $\mathbf{u}_0 \in \mathbb{K}_{G[\mathbf{u}_0]}$ , the quasi-variational inequality*

$$\left\{ \begin{array}{l} \mathbf{u} \in \mathbb{K}_{G[\mathbf{u}]}, \\ \int_0^T \langle \partial_t \mathbf{v}, \mathbf{v} - \mathbf{u} \rangle_p + \int_{Q_T} \alpha \sum_{j=1}^{\ell} \left( \sum_{i=1}^{\ell} |X_i \mathbf{u}|^2 \right)^{\frac{p-2}{2}} X_j \mathbf{u} X_j (\mathbf{v} - \mathbf{u}) \geq \int_{Q_T} f(\mathbf{v} - \mathbf{u}) - \frac{1}{2} \int_{\Omega} |\mathbf{v}(0) - \mathbf{u}_0|^2, \\ \text{for all } \mathbf{v} \in \mathcal{V}_p \text{ such that } \mathbf{v} \in \mathbb{K}_{G[\mathbf{u}]} \end{array} \right.$$

has a weak solution.

## 4 The approximated problem

In order to establish the existence of a solution to the quasi-variational inequality (2.4), we start by proving the existence of the solution to the problem of an approximated system of equations, defined for fixed  $\boldsymbol{\varphi} \in \mathcal{H}$ ,

$\delta \in (0, 1)$  and  $\varepsilon \in (0, 1)$ . With this regularization and penalization of the quasi-variational inequality (2.4) with convex sets  $\mathbb{K}_{G[\varphi]}(t)$ ,  $t \in [0, T]$ , we apply a fixed point argument. Consider the following increasing continuous function  $k_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}_0^+$ :

$$k_\varepsilon(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ e^{\frac{s}{\varepsilon}} - 1 & \text{if } 0 \leq s \leq \frac{1}{\varepsilon}, \\ e^{\frac{1}{\varepsilon^2}} - 1 & \text{if } s \geq \frac{1}{\varepsilon}. \end{cases} \quad (4.1)$$

Observe that the function  $k_{\varepsilon\delta} = \delta + k_\varepsilon$  approximates the maximal monotone graph

$$\bar{k}_\delta(s) \in \begin{cases} \{\delta\} & \text{if } s < 0, \\ [\delta, \infty[ & \text{if } s = 0. \end{cases}$$

We start with an auxiliary lemma.

**Lemma 4.1.** *Let  $\psi$  be a scalar real function defined in  $Q_T$ . Then, for  $p \in (1, \infty)$ , the operator*

$$\mathbf{T}_\varepsilon(x, t, \xi) = k_\varepsilon(|\xi| - \psi(x, t))|\xi|^{p-2}\xi \quad (4.2)$$

is monotone.

*Proof.* To simplify, we omit the argument  $(x, t)$  and we denote  $k_\varepsilon(|\xi| - \psi)$  simply by  $k_\varepsilon(|\xi|)$ . We may assume, without loss of generality, that  $|\xi| \geq |\xi'|$ . Because  $\mathbf{S}(\xi) = |\xi|^{p-2}\xi$  is monotone and  $k_\varepsilon$  is a nonnegative increasing function, we have

$$\begin{aligned} (\mathbf{T}_\varepsilon(\xi) - \mathbf{T}_\varepsilon(\xi')) \cdot (\xi - \xi') &= (k_\varepsilon(|\xi|)\mathbf{S}(\xi) - k_\varepsilon(|\xi'|)\mathbf{S}(\xi')) \cdot (\xi - \xi') \\ &= k_\varepsilon(|\xi'|)(\mathbf{S}(\xi) - \mathbf{S}(\xi')) \cdot (\xi - \xi') + (k_\varepsilon(|\xi|) - k_\varepsilon(|\xi'|))|\xi|^{p-2}\xi \cdot (\xi - \xi') \\ &\geq (k_\varepsilon(|\xi|) - k_\varepsilon(|\xi'|))|\xi|^{p-1}(|\xi| - |\xi'|) \geq 0. \end{aligned} \quad \square$$

**Proposition 4.2.** *Suppose that Assumptions 2.1 to 2.5 are satisfied. By considering functions*

$$\mathbf{f} \in L^2(Q_T)^m, \quad \varphi \in \mathcal{H}, \quad \mathbf{u}_0 \in \mathbb{K}_{G[\varphi]}(0), \quad (4.3)$$

the problem that consists of finding  $\mathbf{u}_{\varepsilon\delta, \varphi}$  such that

$$\left\{ \begin{aligned} &\langle \partial_t \mathbf{u}_{\varepsilon\delta, \varphi}(t), \boldsymbol{\psi} \rangle_p + \int_{\Omega} \mathbf{a}(t, L\mathbf{u}_{\varepsilon\delta, \varphi}(t)) \cdot L\boldsymbol{\psi} + \int_{\Omega} \mathbf{b}(t, \mathbf{u}_{\varepsilon\delta, \varphi}(t)) \cdot \boldsymbol{\psi} \\ &\quad + \int_{\Omega} (\delta + k_\varepsilon(|L\mathbf{u}_{\varepsilon\delta, \varphi}(t)| - G[\varphi](t)))|L\mathbf{u}_{\varepsilon\delta, \varphi}(t)|^{p-2}L\mathbf{u}_{\varepsilon\delta, \varphi}(t) \cdot L\boldsymbol{\psi} \\ &= \int_{\Omega} \mathbf{f}(t) \cdot \boldsymbol{\psi} \quad \text{for all } \boldsymbol{\psi} \in \mathbb{X}_p \text{ for a.e. } t \in (0, T), \\ &\mathbf{u}_{\varepsilon\delta, \varphi}(0) = \mathbf{u}_0 \end{aligned} \right. \quad (4.4)$$

has a unique solution  $\mathbf{u}_{\varepsilon\delta, \varphi} \in \mathcal{V}_p$  with  $\partial_t \mathbf{u}_{\varepsilon\delta, \varphi} \in \mathcal{V}'_p$ , i.e.  $\mathbf{u}_{\varepsilon\delta, \varphi} \in \mathcal{B}_p \subset \mathcal{C}([0, T]; \mathbb{H})$ .

*Proof.* The existence and uniqueness of the solution of problem (4.4) is a consequence of a general result for parabolic quasilinear operators of monotone type (see, for instance, [33, Theorem 8.9, p. 224 or Theorem 8.30, p. 243]). □

**Proposition 4.3.** *Suppose that Assumptions 2.1–2.5 are satisfied. Under assumption (4.3), the solution  $\mathbf{u}_{\varepsilon\delta, \varphi}$  of the problem (4.4) verifies the following a priori estimates:*

$$\|\mathbf{u}_{\varepsilon\delta, \varphi}\|_{L^\infty(0, T; L^2(\Omega)^m)} \leq C, \quad (4.5)$$

$$\|L\mathbf{u}_{\varepsilon\delta, \varphi}\|_{L^p(Q_T)^{\ell}} \leq \frac{C}{\delta^{1/p}}, \quad (4.6)$$

$$\|\partial_t \mathbf{u}_{\varepsilon\delta, \varphi}\|_{(\mathcal{V}'_p)'} \leq \frac{C_\varepsilon}{\delta^{1/p'}},$$

where  $C$  and  $C_\varepsilon$  are positive constants independent of  $\varphi$  and of  $\delta$ , and  $C$  is also independent of  $\varepsilon$ .

*Proof.* Using  $\mathbf{w} = \mathbf{u}_{\varepsilon\delta, \boldsymbol{\varphi}}$  as a test function in (4.4), we get, for a.e.  $t \in (0, T)$ ,

$$\begin{aligned} \langle \partial_t \mathbf{w}(t), \mathbf{w}(t) \rangle_p + \int_{\Omega} \mathbf{a}(t, L\mathbf{w}(t)) \cdot L\mathbf{w}(t) + \int_{\Omega} \mathbf{b}(t, \mathbf{w}(t)) \cdot \mathbf{w}(t) + \int_{\Omega} (\delta + k_{\varepsilon}(|L\mathbf{w}(t)| - G[\boldsymbol{\varphi}](t))) |L\mathbf{w}(t)|^p \\ = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{w}(t). \end{aligned}$$

Set  $Q_t = \Omega \times (0, t)$ . Integrating the last equality between 0 and  $t$ , recalling the monotonicity of  $\mathbf{a}$ ,  $\mathbf{b}$  and of  $\mathbf{T}_{\varepsilon}$  defined in (4.2), and applying the Hölder and Young inequalities to the right-hand side of the above equation, we obtain the inequality

$$\int_{\Omega} |\mathbf{w}(t)|^2 + 2\delta \int_{Q_t} |L\mathbf{w}|^p \leq \|\mathbf{w}\|_{L^2(Q_t)^m}^2 + \|\mathbf{f}\|_{L^2(Q_t)^m}^2 + \int_{\Omega} |\mathbf{u}_0|^2. \quad (4.7)$$

By the Gronwall inequality we conclude that

$$\|\mathbf{w}\|_{L^{\infty}(0, T; L^2(\Omega))} \leq e^T (\|\mathbf{f}\|_{L^2(Q_T)^m}^2 + \|\mathbf{u}_0\|_{L^2(\Omega)^m}^2),$$

and so we proved (4.5). From (4.7) we immediately obtain (4.6).

Next we prove that, given  $\boldsymbol{\psi} \in \mathbb{X}_p$ ,

$$\left| \int_{Q_T} \mathbf{b}(\mathbf{w}) \cdot \boldsymbol{\psi} \right| \leq C \max\{\|\mathbf{w}\|_{L^p(Q_T)}^{p-1}, \|\mathbf{w}\|_{L^{\infty}(0, T; L^2(\Omega))}\} \|\boldsymbol{\psi}\|_{\mathbb{X}_p}, \quad (4.8)$$

with  $C$  being a positive constant. We notice that, by Assumption 2.3,  $\mathbb{X}_p \subset L^2(\Omega)^m$ . So there exists a positive constant  $C$  such that, for all  $\mathbf{v} \in \mathbb{X}_p$ , we have

$$\|\mathbf{v}\|_{L^{2vp}(\Omega)^m} \leq C \|\mathbf{v}\|_{\mathbb{X}_p}.$$

We split the proof in two cases.

(i)  $1 < p < 2$ . Then

$$\left| \int_{Q_T} \mathbf{b}(\mathbf{w}) \cdot \boldsymbol{\psi} \right| \leq b^* \int_0^T \int_{\Omega} |\mathbf{w}(t)| |\boldsymbol{\psi}| \leq b^* \int_0^T \|\mathbf{w}(t)\|_{L^2(\Omega)^m} \|\boldsymbol{\psi}\|_{L^2(\Omega)^m} \leq C b^* T \|\mathbf{w}\|_{L^{\infty}(0, T; L^2(\Omega)^m)} \|\boldsymbol{\psi}\|_{\mathbb{X}_p}.$$

(ii)  $p \geq 2$ . Then

$$\left| \int_{Q_T} \mathbf{b}(\mathbf{w}) \cdot \boldsymbol{\psi} \right| \leq b^* \int_{Q_T} |\mathbf{w}|^{p-1} |\boldsymbol{\psi}| \leq b^* \int_0^T \|\mathbf{w}(t)\|_{L^p(\Omega)^m}^{p-1} \|\boldsymbol{\psi}\|_{L^p(\Omega)^m} \leq C b^* T^{\frac{1}{p}} \|\mathbf{w}\|_{L^p(Q_T)^m}^{p-1} \|\boldsymbol{\psi}\|_{\mathbb{X}_p}.$$

From the first equation of (4.4) we conclude that

$$\begin{aligned} |\langle \partial_t \mathbf{w}(t), \boldsymbol{\psi} \rangle_p| \leq C_1 ((a^* + \delta + e^{\frac{1}{\varepsilon^2}}) \|L\mathbf{w}(t)\|_{L^p(\Omega)^{\ell}}^{p-1} \|L\boldsymbol{\psi}\|_{L^p(\Omega)^{\ell}} + \max\{\|\mathbf{w}\|_{L^p(Q_T)^m}^{p-1}, \|\mathbf{w}\|_{L^{\infty}(0, T; L^2(\Omega)^m)}\}) \|\boldsymbol{\psi}\|_{\mathbb{X}_p} \\ + \|\mathbf{f}(t)\|_{L^2(\Omega)^m} \|\boldsymbol{\psi}\|_{L^2(\Omega)^m}. \end{aligned}$$

Using again Assumption 2.3, we obtain

$$\begin{aligned} \int_0^T |\langle \partial_t \mathbf{w}(t), \boldsymbol{\psi} \rangle_p|^{p'} dt \\ \leq C_2 ((a^* + e^{\frac{1}{\varepsilon^2}})^{p'} \|L\mathbf{w}\|_{L^p(Q_T)^{\ell}}^{p'} + \max\{\|\mathbf{w}\|_{L^p(Q_T)^m}^{p'}, \|\mathbf{w}\|_{L^{\infty}(0, T; L^2(\Omega)^m)}^{p'}\}) + \|\mathbf{f}\|_{L^2(Q_T)^m}^{p'} \|\boldsymbol{\psi}\|_{\mathbb{X}_p}^{p'}, \end{aligned}$$

concluding now easily that

$$\|\partial_t \mathbf{w}(t)\|_{\mathcal{Y}_p'}^{p'} = \int_0^T \sup_{\|\boldsymbol{\psi}\|_{\mathbb{X}_p} \leq 1} |\langle \partial_t \mathbf{w}(t), \boldsymbol{\psi} \rangle_p|^{p'} dt \leq \frac{C_{\varepsilon}}{\delta}.$$

□

**Proposition 4.4.** *Suppose that Assumptions 2.1–2.5 are verified. Assuming also (4.3), define the function  $S : \mathcal{H} \rightarrow \mathcal{Y}_p$  by  $S(\boldsymbol{\varphi}) = \mathbf{u}_{\varepsilon\delta, \boldsymbol{\varphi}}$ , where  $\mathbf{u}_{\varepsilon\delta, \boldsymbol{\varphi}}$  is the unique solution of problem (4.4). Then  $S$  is continuous.*

*Proof.* Let us consider a sequence  $\{\boldsymbol{\varphi}_n\}_n$  converging to  $\boldsymbol{\varphi}$  in  $\mathcal{H}$ . Setting  $\mathbf{w}_n = \mathbf{u}_{\varepsilon\delta, \boldsymbol{\varphi}_n}$  and  $\mathbf{w} = \mathbf{u}_{\varepsilon\delta, \boldsymbol{\varphi}}$ , we need to prove that

$$\mathbf{w}_n \xrightarrow{n} \mathbf{w} \text{ in } \mathcal{Y}_p \quad \text{and} \quad \partial_t \mathbf{w}_n \xrightarrow{n} \partial_t \mathbf{w} \text{ in } \mathcal{Y}'_p.$$

The argument is standard, but we present it here for the sake of completeness. Both functions  $\mathbf{w}_n$  and  $\mathbf{w}$  solve (4.4), so, for any  $\boldsymbol{\psi} \in \mathbb{X}_p$ ,

$$\begin{aligned} & \langle \partial_t(\mathbf{w}_n(t) - \mathbf{w}(t)), \boldsymbol{\psi} \rangle_p + \int_{\Omega} (\mathbf{a}(t, L\mathbf{w}_n(t)) - \mathbf{a}(t, L\mathbf{w}(t))) \cdot L\boldsymbol{\psi} + \int_{\Omega} (\mathbf{b}(t, \mathbf{w}_n(t)) - \mathbf{b}(t, \mathbf{w}(t))) \cdot \boldsymbol{\psi} \\ & + \delta \int_{\Omega} (|L\mathbf{w}_n(t)|^{p-2} L\mathbf{w}_n(t) - |L\mathbf{w}(t)|^{p-2} L\mathbf{w}(t)) \cdot L\boldsymbol{\psi} + \int_{\Omega} (k_{\varepsilon}(|L\mathbf{w}_n(t)| - G[\boldsymbol{\varphi}_n](t)) |L\mathbf{w}_n(t)|^{p-2} L\mathbf{w}_n(t) \\ & - k_{\varepsilon}(|L\mathbf{w}(t)| - G[\boldsymbol{\varphi}](t)) |L\mathbf{w}(t)|^{p-2} L\mathbf{w}(t)) \cdot L\boldsymbol{\psi} = 0. \end{aligned}$$

Replacing  $\boldsymbol{\psi}$  by  $\mathbf{w}_n(t) - \mathbf{w}(t)$  in the last expression and integrating it over  $(0, t)$ , we get

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\mathbf{w}_n(t) - \mathbf{w}(t)|^2 + \int_{Q_t} (\mathbf{a}(L\mathbf{w}_n) - \mathbf{a}(L\mathbf{w})) \cdot L(\mathbf{w}_n - \mathbf{w}) + \int_{Q_t} (\mathbf{b}(\mathbf{w}_n) - \mathbf{b}(\mathbf{w})) \cdot (\mathbf{w}_n - \mathbf{w}) \\ & + \delta \int_{\Omega} (|L\mathbf{w}_n|^{p-2} L\mathbf{w}_n - |L\mathbf{w}|^{p-2} L\mathbf{w}) \cdot L(\mathbf{w}_n - \mathbf{w}) \\ & + \int_{Q_t} (k_{\varepsilon}(|L\mathbf{w}_n| - G[\boldsymbol{\varphi}_n]) |L\mathbf{w}_n|^{p-2} L\mathbf{w}_n - k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}]) |L\mathbf{w}|^{p-2} L\mathbf{w}) \cdot L(\mathbf{w}_n - \mathbf{w}) \\ & = \int_{Q_t} (k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}]) - k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}_n])) |L\mathbf{w}|^{p-2} L\mathbf{w} \cdot L(\mathbf{w}_n - \mathbf{w}). \end{aligned} \quad (4.9)$$

Using the monotonicity of  $\mathbf{a}$ ,  $\mathbf{b}$  and the operator  $T_{\varepsilon}$  defined in (4.2), we can neglect the second, third and fifth terms of the inequality above.

In the case  $p \geq 2$ , we obtain, applying the Hölder and Young inequalities and denoting by  $D_p$  the constant related to the strongly monotone term in  $\delta$  (see (2.12)),

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\mathbf{w}_n(t) - \mathbf{w}(t)|^2 + \delta D_p \int_{Q_t} |L(\mathbf{w}_n - \mathbf{w})|^p \\ & \leq \int_{Q_t} |k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}]) - k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}_n])| |L\mathbf{w}|^{p-1} |L(\mathbf{w}_n - \mathbf{w})| \\ & \leq \frac{C_1}{\delta^{p-1}} \int_{Q_t} |k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}]) - k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}_n])|^{p'} |L\mathbf{w}|^p + \frac{\delta}{2} D_p \int_{Q_t} |L(\mathbf{w}_n - \mathbf{w})|^p, \end{aligned}$$

and therefore we get

$$\begin{aligned} & \|\mathbf{w}_n - \mathbf{w}\|_{L^{\infty}(0, T; L^2(\Omega)^m)}^2 + \delta D_p \|L(\mathbf{w}_n - \mathbf{w})\|_{L^p(Q_t)^{\varepsilon}}^p \\ & \leq \frac{2C_1}{\delta^{p-1}} \int_{Q_t} |k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}]) - k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}_n])|^{p'} |L\mathbf{w}|^p. \end{aligned} \quad (4.10)$$

Consider now the case  $1 < p < 2$ . From (4.9) we get again

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\mathbf{w}_n(t) - \mathbf{w}(t)|^2 + \delta \int_{Q_t} (|L\mathbf{w}_n|^{p-2} L\mathbf{w}_n - |L\mathbf{w}|^{p-2} L\mathbf{w}) \cdot L(\mathbf{w}_n - \mathbf{w}) \\ & \leq \int_{Q_t} |k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}]) - k_{\varepsilon}(|L\mathbf{w}| - G[\boldsymbol{\varphi}_n])| |L\mathbf{w}|^{p-1} |L(\mathbf{w}_n - \mathbf{w})|, \end{aligned}$$

and, using also the coercive condition on  $\delta$  (see (2.12)) and the Hölder inverse inequality, we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\mathbf{w}_n(t) - \mathbf{w}(t)|^2 + \delta D_p(\|\mathbf{L}\mathbf{w}_n\|_{L^p(Q_T)^\epsilon}^p + \|\mathbf{L}\mathbf{w}\|_{L^p(Q_T)^\epsilon}^p)^{\frac{p-2}{p}} \|\mathbf{L}(\mathbf{w}_n - \mathbf{w})\|_{L^p(Q_T)^\epsilon}^2 \\ & \leq \int_{Q_t} |k_\epsilon(|\mathbf{L}\mathbf{w}| - G[\boldsymbol{\varphi}]) - k_\epsilon(|\mathbf{L}\mathbf{w}_n| - G[\boldsymbol{\varphi}_n])| |\mathbf{L}\mathbf{w}|^{p-1} |\mathbf{L}(\mathbf{w}_n - \mathbf{w})|. \end{aligned}$$

Recalling, by (4.6),

$$\|\mathbf{L}\mathbf{w}_n\|_{L^p(Q_T)^\epsilon}^p \leq \frac{C^p}{\delta}, \quad \|\mathbf{L}\mathbf{w}\|_{L^p(Q_T)^\epsilon}^p \leq \frac{C^p}{\delta}$$

and applying the Hölder and Young inequalities to the right-hand side, we obtain

$$\begin{aligned} & \|\mathbf{w}_n - \mathbf{w}\|_{L^\infty(0,T;L^2(\Omega)^m)}^2 + \delta^{\frac{2}{p}} C_2 \|\mathbf{L}(\mathbf{w}_n - \mathbf{w})\|_{L^p(Q_T)^\epsilon}^2 \\ & \leq \frac{1}{2C_2\delta^{\frac{2}{p}}} \left( \int_{Q_T} |k_\epsilon(|\mathbf{L}\mathbf{w}| - G[\boldsymbol{\varphi}]) - k_\epsilon(|\mathbf{L}\mathbf{w}_n| - G[\boldsymbol{\varphi}_n])|^{p'} |\mathbf{L}\mathbf{w}|^p \right)^{\frac{2}{p'}} + \frac{C_2\delta^{\frac{2}{p}}}{2} \|\mathbf{L}(\mathbf{w}_n - \mathbf{w})\|_{L^p(Q_T)^\epsilon}^2, \end{aligned}$$

and so

$$\begin{aligned} & \|\mathbf{w}_n - \mathbf{w}\|_{L^\infty(0,T;L^2(\Omega)^m)}^2 + \frac{C_2\delta^{\frac{2}{p}}}{2} \|\mathbf{L}(\mathbf{w}_n - \mathbf{w})\|_{L^p(Q_T)^\epsilon}^2 \\ & \leq \frac{1}{2C_2\delta^{\frac{2}{p}}} \left( \int_{Q_T} |k_\epsilon(|\mathbf{L}\mathbf{w}| - G[\boldsymbol{\varphi}]) - k_\epsilon(|\mathbf{L}\mathbf{w}_n| - G[\boldsymbol{\varphi}_n])|^{p'} |\mathbf{L}\mathbf{w}|^p \right)^{\frac{2}{p'}}. \end{aligned} \quad (4.11)$$

Observe now that we have

$$|k_\epsilon(|\mathbf{L}\mathbf{w}| - G[\boldsymbol{\varphi}]) - k_\epsilon(|\mathbf{L}\mathbf{w}_n| - G[\boldsymbol{\varphi}_n])|^{p'} |\mathbf{L}\mathbf{w}|^p \leq (2e^{\frac{1}{\epsilon^2}})^{p'} |\mathbf{L}\mathbf{w}|^p$$

a.e. in  $Q_T$ , and  $k_\epsilon$  is a continuous function. By recalling that  $\boldsymbol{\varphi}_n \xrightarrow{n} \boldsymbol{\varphi}$  in  $\mathcal{H}$ , Assumption 2.6 implies that

$$G[\boldsymbol{\varphi}_n] \xrightarrow{n} G[\boldsymbol{\varphi}]$$

in  $L^1(Q_T)$ . Hence, at least for a subsequence,

$$G[\boldsymbol{\varphi}_n] \xrightarrow{n} G[\boldsymbol{\varphi}] \quad \text{a.e. in } Q_T$$

and, by the dominated convergence theorem,

$$k_\epsilon(|\mathbf{L}\mathbf{w}_n| - G[\boldsymbol{\varphi}_n]) \xrightarrow{n} k_\epsilon(|\mathbf{L}\mathbf{w}| - G[\boldsymbol{\varphi}]) \quad \text{in } L^{p'}(Q_T).$$

Therefore, the right-hand sides of (4.10) and (4.11) converge to zero a.e. when  $n \rightarrow \infty$ .

By definition,

$$\begin{aligned} \|\partial_t(\mathbf{w}_n - \mathbf{w})\|_{\mathcal{Y}'}^{p'} &= \int_0^T \left( \sup_{\|\boldsymbol{\psi}\|_{\mathbb{X}_p} \leq 1} \langle \partial_t(\mathbf{w}_n(t) - \mathbf{w}(t)), \boldsymbol{\psi} \rangle_p \right)^{p'} \\ &= \int_0^T \left( \sup_{\|\boldsymbol{\psi}\|_{\mathbb{X}_p} \leq 1} \left( \int_{\Omega} (\mathbf{a}(t, \mathbf{L}\mathbf{w}(t)) - \mathbf{a}(t, \mathbf{L}\mathbf{w}_n(t))) \cdot \mathbf{L}\boldsymbol{\psi} + \int_{\Omega} (\mathbf{b}(t, \mathbf{w}(t)) - \mathbf{b}(t, \mathbf{w}_n(t))) \cdot \boldsymbol{\psi} \right. \right. \\ & \quad + \delta \int_{\Omega} (|\mathbf{L}\mathbf{w}(t)|^{p-2} \mathbf{L}\mathbf{w}(t) - |\mathbf{L}\mathbf{w}_n(t)|^{p-2} \mathbf{L}\mathbf{w}_n(t)) \cdot \mathbf{L}\boldsymbol{\psi} \\ & \quad + \int_{\Omega} k_\epsilon(|\mathbf{L}\mathbf{w}(t)| - G[\boldsymbol{\varphi}](t)) |\mathbf{L}\mathbf{w}(t)|^{p-2} \mathbf{L}\mathbf{w}(t) \cdot \mathbf{L}\boldsymbol{\psi} \\ & \quad \left. \left. - \int_{\Omega} k_\epsilon(|\mathbf{L}\mathbf{w}_n(t)| - G[\boldsymbol{\varphi}_n](t)) |\mathbf{L}\mathbf{w}_n(t)|^{p-2} \mathbf{L}\mathbf{w}_n(t) \cdot \mathbf{L}\boldsymbol{\psi} \right) \right)^{p'}. \end{aligned}$$

But

$$\begin{aligned} \left( \int_{\Omega} (\mathbf{b}(t, \mathbf{w}(t)) - \mathbf{b}(t, \mathbf{w}_n(t))) \cdot \boldsymbol{\psi} \right)^{p'} &\leq (\|(\mathbf{b}(t, \mathbf{w}(t)) - \mathbf{b}(t, \mathbf{w}_n(t)))\|_{L^2(\Omega)^m} \|\boldsymbol{\psi}\|_{L^2(\Omega)^m})^{p'} \\ &\leq C(\|(\mathbf{b}(\mathbf{w}) - \mathbf{b}(\mathbf{w}_n))\|_{L^\infty(0, T; (L^2(\Omega)^m))} \|\boldsymbol{\psi}\|_{\mathbb{X}_p})^{p'}. \end{aligned}$$

Applying the Hölder inequality, we conclude that

$$\begin{aligned} \|\partial_t(\mathbf{w}_n - \mathbf{w})\|_{\mathcal{Y}_p'} &\leq C \left( \left( \int_{Q_T} |\boldsymbol{\alpha}(L\mathbf{w}) - \boldsymbol{\alpha}(L\mathbf{w}_n)|^{p'} \right)^{\frac{1}{p'}} + \|(\mathbf{b}(\mathbf{w}) - \mathbf{b}(\mathbf{w}_n))\|_{L^\infty(0, T; (L^2(\Omega)^m))} \right. \\ &\quad \left. + \delta \left( \int_{Q_T} \left| |\mathbf{L}\mathbf{w}_n|^{p-2} \mathbf{L}\mathbf{w}_n - |\mathbf{L}\mathbf{w}|^{p-2} \mathbf{L}\mathbf{w} \right|^{p'} \right)^{\frac{1}{p'}} \right. \\ &\quad \left. + \left( \int_{Q_T} |k_\varepsilon(|\mathbf{L}\mathbf{w}| - G[\boldsymbol{\varphi}])|\mathbf{L}\mathbf{w}|^{p-2} \mathbf{L}\mathbf{w} - k_\varepsilon(|\mathbf{L}\mathbf{w}_n| - G[\boldsymbol{\varphi}_n])|\mathbf{L}\mathbf{w}_n|^{p-2} \mathbf{L}\mathbf{w}_n|^{p'} \right)^{\frac{1}{p'}} \right), \end{aligned}$$

and, arguing as before, we conclude the proof.  $\square$

**Theorem 4.5.** *Suppose that Assumptions 2.1–2.6 and (4.3) are verified. Let  $i$  be the inclusion of  $\mathcal{Y}_p$  into  $\mathcal{H}$  and let  $S : \mathcal{H} \rightarrow \mathcal{Y}_p$  be the function defined in Proposition 4.4. Then the function  $i \circ S$  has a fixed point in  $\mathcal{H}$ . This fixed point solves the problem that consists of finding  $\mathbf{u}_{\varepsilon\delta} \in \mathcal{Y}_p$  such that*

$$\begin{cases} \langle \partial_t \mathbf{u}_{\varepsilon\delta}(t), \boldsymbol{\psi} \rangle_p + \int_{\Omega} \boldsymbol{\alpha}(t, \mathbf{L}\mathbf{u}_{\varepsilon\delta}(t)) \cdot \mathbf{L}\boldsymbol{\psi} + \int_{\Omega} \mathbf{b}(t, \mathbf{u}_{\varepsilon\delta}(t)) \cdot \boldsymbol{\psi} + \delta \int_{\Omega} |\mathbf{L}\mathbf{u}_{\varepsilon\delta}(t)|^{p-2} \mathbf{L}\mathbf{u}_{\varepsilon\delta}(t) \cdot \mathbf{L}\boldsymbol{\psi} \\ \quad + \int_{\Omega} k_\varepsilon(|\mathbf{L}\mathbf{u}_{\varepsilon\delta}(t)| - G[\mathbf{u}_{\varepsilon\delta}(t)]) |\mathbf{L}\mathbf{u}_{\varepsilon\delta}(t)|^{p-2} \mathbf{L}\mathbf{u}_{\varepsilon\delta}(t) \cdot \mathbf{L}\boldsymbol{\psi} \\ = \int_{\Omega} \mathbf{f}(t) \cdot \boldsymbol{\psi} \quad \text{for all } \boldsymbol{\psi} \in \mathbb{X}_p \\ \mathbf{u}_{\varepsilon\delta}(0) = \mathbf{u}_0. \end{cases} \quad (4.12)$$

*Proof.* We use the Schauder fixed point theorem. We already proved the continuity of  $S$ . By Assumption 2.5 and the Aubin–Lions lemma the embedding  $\mathcal{Y}_p \hookrightarrow \mathcal{H}$  is compact for  $1 < p < \infty$ , and so  $i \circ S$  is completely continuous as a map of  $\mathcal{H}$  into itself. By Proposition 4.3, given  $\boldsymbol{\varphi} \in \mathcal{H}$ , we have

$$\|\mathbf{u}_{\varepsilon\delta, \boldsymbol{\varphi}}\|_{\mathcal{Y}_p} \leq \frac{C}{\delta^{1/p}},$$

where  $C$  is a constant independent of  $\boldsymbol{\varphi}$  and  $\delta$  (it may depend on  $\varepsilon$ ). Because  $i$  is continuous, there exists  $C_1$  such that  $\|\mathbf{v}\|_{\mathcal{H}} \leq C_1 \|\mathbf{v}\|_{\mathcal{Y}_p}$ , and we get

$$\|i \circ S(\boldsymbol{\varphi})\|_{\mathcal{H}} \leq CC_1.$$

Then the image of  $i \circ S$  is bounded, so we may apply the Schauder fixed point theorem, obtaining immediately the conclusion of the existence of a  $\mathbf{u}_{\varepsilon\delta} = i \circ S(\mathbf{u}_{\varepsilon\delta})$  in  $\mathcal{Y}_p$ .  $\square$

## 5 Weak solutions of the quasi-variational inequality

In this section, we prove the existence of a solution of the quasi-variational inequality (2.4) by taking suitable subsequences of solutions of (4.12) first as  $\varepsilon \rightarrow 0$  and then as  $\delta \rightarrow 0$ .

Firstly we collect the a priori estimates for the solution  $\mathbf{u}_{\varepsilon\delta}$  of problem (4.12) which are independent of  $\varepsilon$ .

**Proposition 5.1.** *Suppose that Assumptions 2.1–2.6 are verified. Assume that  $\mathbf{f} \in L^2(Q_T)^m$  and  $\mathbf{u}_0 \in \mathbb{K}_{[G(\mathbf{u}_0)]}$ . Let  $\mathbf{u}_{\varepsilon\delta}$  be a solution of the approximated problem (4.12). Then there exists a positive constant  $C$  independent of  $\varepsilon$  and  $\delta$  such that*

$$\|\mathbf{u}_{\varepsilon\delta}\|_{L^\infty(0,T;L^2(\Omega)^m)} \leq C, \quad (5.1)$$

$$\|\mathbf{u}_{\varepsilon\delta}\|_{\mathcal{V}_p} \leq \frac{C}{\delta^{1/p}}, \quad (5.2)$$

$$\|\mathbf{a}(\mathbf{L}\mathbf{u}_{\varepsilon\delta})\|_{\mathcal{V}'_p} \leq \frac{C}{\delta^{1/p'}}, \quad (5.3)$$

$$\|\mathbf{b}(\mathbf{u}_{\varepsilon\delta})\|_{\mathcal{V}'_p} \leq \frac{C}{\delta^{1/p'}}, \quad (5.4)$$

$$\|k_\varepsilon(|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}])\|_{L^1(Q_T)} \leq C. \quad (5.5)$$

*Proof.* The first two estimates are direct consequences of inequalities (4.5) and (4.6), respectively, taking  $\boldsymbol{\varphi} = \mathbf{u}_{\varepsilon\delta}$ .

Given  $\boldsymbol{\psi} \in \mathcal{V}_p$ , we have

$$\left| \int_{Q_T} \mathbf{a}(\mathbf{L}\mathbf{u}_{\varepsilon\delta}) \cdot \mathbf{L}\boldsymbol{\psi} \right| \leq a^* \int_{Q_T} |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-1} |\mathbf{L}\boldsymbol{\psi}| \leq a^* \|\mathbf{L}\mathbf{u}_{\varepsilon\delta}\|_{L^p(Q_T)}^{p-1} \|\mathbf{L}\boldsymbol{\psi}\|_{L^p(Q_T)} \leq a^* \frac{C^{p-1}}{\delta^{1/p'}} \|\mathbf{L}\boldsymbol{\psi}\|_{\mathcal{V}_p},$$

proving (5.3).

From (4.8) we have

$$\|\mathbf{b}(\mathbf{w})\|_{\mathcal{V}'_p} \leq C \max\{\|\mathbf{w}\|_{L^p(Q_T)^m}^{p-1}, \|\mathbf{w}\|_{L^\infty(0,T;L^2(\Omega)^m)}\}.$$

But  $\|\mathbf{w}\|_{L^\infty(0,T;L^2(\Omega)^m)}$  is uniformly bounded by (5.1), and

$$\|\mathbf{w}\|_{L^p(Q_T)^m}^{p-1} \leq C_1 \|\mathbf{L}\mathbf{w}\|_{L^p(Q_T)^m}^{p-1} \leq \frac{C_2}{\delta^{1/p'}}$$

by (5.2).

Choosing  $\mathbf{u}_{\varepsilon\delta}$  as test function in (4.12), and integrating between 0 and  $t$ , we get

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\mathbf{u}_{\varepsilon\delta}(t)|^2 + \int_{Q_t} \mathbf{a}(\mathbf{L}\mathbf{u}_{\varepsilon\delta}) \cdot \mathbf{L}\mathbf{u}_{\varepsilon\delta} + \int_{Q_t} \mathbf{b}(\mathbf{u}_{\varepsilon\delta}) \cdot \mathbf{u}_{\varepsilon\delta} + \delta \int_{Q_t} |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^p + \int_{Q_t} k_\varepsilon(|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^p \\ &= \int_{Q_t} \mathbf{f} \cdot \mathbf{u}_{\varepsilon\delta} + \frac{1}{2} \int_{\Omega} \mathbf{u}_0^2. \end{aligned}$$

Therefore, because  $\mathbf{a}$  and  $\mathbf{b}$  are monotone,  $\mathbf{a}(\mathbf{0}) = \mathbf{0}$ ,  $\mathbf{b}(\mathbf{0}) = \mathbf{0}$  and by using the Gronwall inequality, we obtain

$$\int_{Q_t} k_\varepsilon(|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^p \leq C_1, \quad (5.6)$$

where  $C_1$  is a constant independent of  $\varepsilon$  and  $\delta$ . As  $G[\mathbf{u}_{\varepsilon\delta}] \geq g_* > 0$  and  $k_\varepsilon \equiv 0$  in  $\{|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| \leq G[\mathbf{u}_{\varepsilon\delta}]\}$ , we obtain

$$\int_{Q_T} k_\varepsilon(|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) = \int_{\{|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| > G[\mathbf{u}_{\varepsilon\delta}]\}} k_\varepsilon(|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) \leq \int_{\{|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| > G[\mathbf{u}_{\varepsilon\delta}]\}} k_\varepsilon(|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) \frac{|\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^p}{(g_*)^p}.$$

Using (5.6), we obtain

$$\|k_\varepsilon(|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}])\|_{L^1(Q_T)} \leq \frac{C_1}{(g_*)^p},$$

concluding the proof of (5.5).  $\square$

**Lemma 5.2.** *Let  $\mathbf{u}_{\varepsilon\delta}$  be a solution of the approximated problem (4.12). If  $\mathbf{u}_\delta$  is the weak limit of a subsequence of  $\{\mathbf{u}_{\varepsilon\delta}\}_\varepsilon$  when  $\varepsilon \rightarrow 0$ , then  $\mathbf{u}_\delta \in \mathbb{K}_{G[\mathbf{u}_\delta]}$ .*

*Proof.* To prove that  $\mathbf{u}_\delta$  belongs to the convex set  $\mathbb{K}_{G[\mathbf{u}_\delta]}$ , we use arguments as in [27], which we adapt to our problem. We split  $Q_T$  in three sets:

$$\begin{cases} A_{\varepsilon\delta} = \{(x, t) \in Q_T : |\mathbf{L}\mathbf{u}_{\varepsilon\delta}(x, t) - G[\mathbf{u}_{\varepsilon\delta}](x, t) < \sqrt{\varepsilon}\}, \\ B_{\varepsilon\delta} = \{(x, t) \in Q_T : \sqrt{\varepsilon} \leq |\mathbf{L}\mathbf{u}_{\varepsilon\delta}(x, t) - G[\mathbf{u}_{\varepsilon\delta}](x, t) < \frac{1}{\varepsilon}\}, \\ C_{\varepsilon\delta} = \{(x, t) \in Q_T : |\mathbf{L}\mathbf{u}_{\varepsilon\delta}(x, t) - G[\mathbf{u}_{\varepsilon\delta}](x, t) \geq \frac{1}{\varepsilon}\}. \end{cases} \quad (5.7)$$

We recall that, by Assumption 2.6, the operator  $G$  is compact. So, as a subsequence of  $\{\mathbf{u}_{\varepsilon\delta}\}_\varepsilon$  (still denoted by  $\{\mathbf{u}_{\varepsilon\delta}\}_\varepsilon$ ) converges weakly to  $\mathbf{u}_\delta$  in  $\mathcal{Y}_p$ , we obtain that  $G[\mathbf{u}_{\varepsilon\delta}]$  converges to  $G[\mathbf{u}_\delta]$  strongly in  $\mathcal{C}([0, T]; L^\infty(\Omega))$  and

$$\begin{aligned} \int_{Q_T} (|\mathbf{L}\mathbf{u}_\delta| - G[\mathbf{u}_\delta])^+ &= \int_{Q_T} \lim_{\varepsilon \rightarrow 0} \left( (|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) \wedge \frac{1}{\varepsilon} \vee \sqrt{\varepsilon} \right) \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_{Q_T} \left( (|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) \wedge \frac{1}{\varepsilon} \vee \sqrt{\varepsilon} \right) \\ &= \lim_{\varepsilon \rightarrow 0} \left( \int_{A_{\varepsilon\delta}} \sqrt{\varepsilon} + \int_{B_{\varepsilon\delta}} (|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) + \int_{C_{\varepsilon\delta}} \frac{1}{\varepsilon} \right). \end{aligned}$$

We observe that

$$\begin{aligned} \int_{A_{\varepsilon\delta}} \sqrt{\varepsilon} &\leq \sqrt{\varepsilon} |Q_T| \xrightarrow{\varepsilon \rightarrow 0} 0, \\ |B_{\varepsilon\delta}| &= \int_{B_{\varepsilon\delta}} 1 \leq \int_{Q_T} \frac{k_\varepsilon (|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) + 1}{e^{1/\sqrt{\varepsilon}}} \leq \frac{C}{e^{1/\sqrt{\varepsilon}}} \xrightarrow{\varepsilon \rightarrow 0} 0 \\ \int_{C_{\varepsilon\delta}} \frac{1}{\varepsilon} &\leq \frac{1}{\varepsilon} \int_{Q_T} \frac{k_\varepsilon (|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}])}{e^{1/\varepsilon^2} - 1} \leq \frac{C}{\varepsilon(e^{1/\varepsilon^2} - 1)} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

Besides,

$$\int_{B_{\varepsilon\delta}} (|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) \leq \| |\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}] \|_{L^p(Q_T)} |B_{\varepsilon\delta}|^{1/p} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

So,

$$\int_{Q_T} (|\mathbf{L}\mathbf{u}_\delta| - G[\mathbf{u}_\delta])^+ = 0,$$

which means that

$$|\mathbf{L}\mathbf{u}_\delta| \leq G[\mathbf{u}_\delta] \quad \text{a.e. in } Q_T. \quad \square$$

**Lemma 5.3.** Let  $\mathbf{v} \in \mathcal{Y}_p = L^p(0, T; \mathbb{X}_p)$  be such that  $\mathbf{v} \in \mathbb{K}_{G[\mathbf{v}]}$ , and let  $\mathbf{z} \in \mathbb{K}_{G[\mathbf{v}](0)}$ . Then there exists a regularizing sequence  $\{\mathbf{v}_n\}_n$  and a sequence of scalar functions  $\{g_n\}_n$  with the following properties:

- (i)  $\mathbf{v}_n \in L^\infty(0, T; \mathbb{X}_p)$  and  $\partial_t \mathbf{v}_n \in L^\infty(0, T; \mathbb{X}_p)$ .
- (ii)  $\mathbf{v}_n \xrightarrow{n} \mathbf{v}$  strongly in  $\mathcal{Y}_p$ .
- (iii)  $\lim_n \int_0^T \langle \partial_t \mathbf{v}_n, \mathbf{v}_n - \mathbf{v} \rangle_p \leq 0$ .
- (iv)  $|\mathbf{L}\mathbf{v}_n| \leq g_n$ , where  $g_n \in \mathcal{C}([0, T]; L^\infty(\Omega))$  and  $g_n \xrightarrow{n} G[\mathbf{v}]$  in  $\mathcal{C}([0, T]; L^\infty(\Omega))$ .

*Proof.* Let  $\mathbf{v}_n$  be the unique solution of the ordinary differential equation  $\mathbf{v}_n + \frac{1}{n} \partial_t \mathbf{v}_n = \mathbf{v}$  with  $\mathbf{v}_n(0) = \mathbf{z}$ . The function  $\mathbf{v}_n$  has the following expression:

$$\mathbf{v}_n(x, t) = e^{-nt} \int_0^t \mathbf{v}(x, \tau) n e^{n\tau} d\tau + e^{-nt} \mathbf{z}(x)$$

and it is well known that it satisfies (i), (ii) and (iii) (see [23, p. 274] or [33, p. 206]). Therefore, it follows

$$\begin{aligned} |L\mathbf{v}_n(x, t)| &\leq e^{-nt} \int_0^t |L\mathbf{v}(x, \tau)| n e^{n\tau} d\tau + e^{-nt} |L\mathbf{z}(x)| \\ &\leq e^{-nt} \int_0^t G[\mathbf{v}](x, \tau) n e^{n\tau} d\tau + e^{-nt} G[\mathbf{v}](x, 0) \quad \text{for a.e. } (x, t) \in Q_T. \end{aligned}$$

Defining

$$g_n(x, t) = e^{-nt} \int_0^t G[\mathbf{v}](x, \tau) n e^{n\tau} d\tau + e^{-nt} G[\mathbf{v}](x, 0) = \int_{-\infty}^0 \widetilde{G}[\mathbf{v}]\left(x, \frac{s}{n} + t\right) e^s ds,$$

where  $\widetilde{G}[\mathbf{v}](x, t)$  denotes the extension of  $G[\mathbf{v}] \in \mathcal{C}([0, T]; L^\infty(\Omega))$  by  $G[\mathbf{v}](x, 0)$  for  $t < 0$ , we have

$$|g_n(x, t) - G[\mathbf{v}](x, t)| \leq \int_{-\infty}^0 |\widetilde{G}[\mathbf{v}]\left(x, \frac{s}{n} + t\right) - G[\mathbf{v}](x, t)| e^s ds$$

and, by the uniform continuity of  $G[\mathbf{v}](t)$  in  $[0, T]$  with values in  $L^\infty(\Omega)$ , we also have the uniform convergence of  $g_n \xrightarrow{n} G[\mathbf{v}]$  in  $\mathcal{C}([0, T]; L^\infty(\Omega))$ , concluding (iv).  $\square$

*Proof of Theorem 2.7.* The boundedness of  $\{\mathbf{u}_{\varepsilon\delta}\}_\varepsilon$  in  $L^\infty(0, T; L^2(\Omega)^m) \cap \mathcal{V}_p$  implies that there exists a subsequence, still denoted by  $\{\mathbf{u}_{\varepsilon\delta}\}_\varepsilon$ , converging weakly-\* to a function  $\mathbf{u}_\delta$  when  $\varepsilon \rightarrow 0$ , in  $L^\infty(0, T; L^2(\Omega)^m) \cap \mathcal{V}_p$ . So, by Lemma 5.2,  $\mathbf{u}_\delta \in \mathbb{K}_{G[\mathbf{u}_\delta]}$  and we may extract a subsequence of  $\{\mathbf{u}_\delta\}_\delta$  converging weakly-\* to some  $\mathbf{u}$  in  $L^\infty(0, T; L^2(\Omega)^m) \cap \mathcal{V}_p$  when  $\delta \rightarrow 0$ .

Observe that, for any measurable set  $\omega \subset Q_T$ ,

$$\int_\omega |L\mathbf{u}| \leq \liminf_{\delta \rightarrow 0} \int_\omega |L\mathbf{u}_\delta| \leq \int_\omega \lim_{\delta \rightarrow 0} G[\mathbf{u}_\delta] \leq \int_\omega G[\mathbf{u}],$$

using Assumption 2.6. Consequently,

$$|L\mathbf{u}| \leq G[\mathbf{u}] \quad \text{a.e. in } Q_T,$$

and so  $\mathbf{u} \in \mathbb{K}_{G[\mathbf{u}]}$ .

**Step 1: The limit when  $\varepsilon \rightarrow 0$ .** From estimates (5.3), (5.4) and (5.2) in Proposition 5.1 there exist  $\chi_\delta \in \mathcal{V}'_p$ ,  $\Upsilon_\delta \in \mathcal{V}'_p$  and  $\Lambda_\delta \in L^{p'}(Q_T)^\ell$  such that for subsequences,

$$\begin{aligned} \mathbf{a}(L\mathbf{u}_{\varepsilon\delta}) &\xrightarrow{\varepsilon \rightarrow 0} \chi_\delta \quad \text{in } \mathcal{V}'_p \text{ weak,} \\ \mathbf{b}(\mathbf{u}_{\varepsilon\delta}) &\xrightarrow{\varepsilon \rightarrow 0} \Upsilon_\delta \quad \text{in } \mathcal{V}'_p \text{ weak.} \\ |L\mathbf{u}_{\varepsilon\delta}|^{p-2} L\mathbf{u}_{\varepsilon\delta} &\xrightarrow{\varepsilon \rightarrow 0} \Lambda_\delta \quad \text{in } L^{p'}(Q_T)^\ell \text{ weak.} \end{aligned} \tag{5.8}$$

Define the operator  $\mathcal{A}_\delta : \mathcal{V}_p \rightarrow \mathcal{V}'_p$  by

$$\langle \mathcal{A}_\delta \mathbf{v}, \mathbf{w} \rangle = \int_{Q_T} (\mathbf{a}(L\mathbf{v}) + \delta |L\mathbf{v}|^{-2} L\mathbf{v}) \cdot L\mathbf{w} + \int_{Q_T} \mathbf{b}(\mathbf{v}) \cdot \mathbf{w}.$$

Given  $\mathbf{v}$  belonging to the space  $\mathcal{B}_p$  defined in (2.2), we have

$$\begin{aligned} \int_0^T \langle \partial_t \mathbf{u}_{\varepsilon\delta}, \mathbf{v} - \mathbf{u}_{\varepsilon\delta} \rangle_p &= -\frac{1}{2} \int_\Omega |\mathbf{u}_{\varepsilon\delta}(T) - \mathbf{v}(T)|^2 + \frac{1}{2} \int_\Omega |\mathbf{u}_0 - \mathbf{v}(0)|^2 + \int_0^T \langle \partial_t \mathbf{v}, \mathbf{v} - \mathbf{u}_{\varepsilon\delta} \rangle_p \\ &\leq \frac{1}{2} \int_\Omega |\mathbf{u}_0 - \mathbf{v}(0)|^2 + \int_0^T \langle \partial_t \mathbf{v}, \mathbf{v} - \mathbf{u}_{\varepsilon\delta} \rangle_p. \end{aligned}$$

From now on, we denote  $k_\varepsilon(|L\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}])$  simply by  $k_\varepsilon$ , with no risk of confusion.

Using  $\mathbf{u}_{\varepsilon\delta} - \mathbf{v}$  as test function in (4.12) and integrating between 0 and  $T$ , we obtain

$$\begin{aligned} \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} \rangle &\leq \frac{1}{2} \int_{\Omega} |\mathbf{u}_0 - \mathbf{v}(0)|^2 + \int_0^T \langle \partial_t \mathbf{v}, \mathbf{v} - \mathbf{u}_{\varepsilon\delta} \rangle_p + \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{v} \rangle \\ &\quad + \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-2} \mathbf{L}\mathbf{u}_{\varepsilon\delta} \cdot L(\mathbf{v} - \mathbf{u}_{\varepsilon\delta}) - \int_{Q_T} \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}_{\varepsilon\delta}). \end{aligned} \quad (5.9)$$

Hence, for all  $\mathbf{v} \in \mathcal{B}_p$ ,

$$\begin{aligned} \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle &\leq \frac{1}{2} \int_{\Omega} |\mathbf{u}_0 - \mathbf{v}(0)|^2 + \int_0^T \langle \partial_t \mathbf{v}, \mathbf{v} - \mathbf{u}_{\varepsilon\delta} \rangle_p + \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{v} - \mathbf{u} \rangle \\ &\quad + \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-2} \mathbf{L}\mathbf{u}_{\varepsilon\delta} \cdot L(\mathbf{v} - \mathbf{u}_{\varepsilon\delta}) - \int_{Q_T} \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}_{\varepsilon\delta}). \end{aligned} \quad (5.10)$$

Let  $\mathbf{u}_n$  be the regularizing sequence of  $\mathbf{u}$  defined in the previous lemma. Using  $\mathbf{u}_n$  as test function in (5.10), we get

$$\begin{aligned} \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle &\leq \int_0^T \langle \partial_t \mathbf{u}_n, \mathbf{u}_n - \mathbf{u}_{\varepsilon\delta} \rangle_p + \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_n - \mathbf{u} \rangle \\ &\quad + \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-2} \mathbf{L}\mathbf{u}_{\varepsilon\delta} \cdot L(\mathbf{u}_n - \mathbf{u}_{\varepsilon\delta}) - \int_{Q_T} \mathbf{f} \cdot (\mathbf{u}_n - \mathbf{u}_{\varepsilon\delta}). \end{aligned} \quad (5.11)$$

But

$$\begin{aligned} \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-2} \mathbf{L}\mathbf{u}_{\varepsilon\delta} \cdot L(\mathbf{u}_n - \mathbf{u}_{\varepsilon\delta}) &\leq \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-1} (|\mathbf{L}\mathbf{u}_n| - |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|) \\ &= \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-1} (|\mathbf{L}\mathbf{u}_n| - G[\mathbf{u}_{\varepsilon\delta}]) + \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-1} (G[\mathbf{u}_{\varepsilon\delta}] - |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|) \\ &\leq \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-1} (|\mathbf{L}\mathbf{u}_n| - G[\mathbf{u}_{\varepsilon\delta}]). \end{aligned} \quad (5.12)$$

In fact, the term  $k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-1} (G[\mathbf{u}_{\varepsilon\delta}] - |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|)$  is less than or equal to zero because when  $|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| < G[\mathbf{u}_{\varepsilon\delta}]$ , then  $k_\varepsilon (|\mathbf{L}\mathbf{u}_{\varepsilon\delta}| - G[\mathbf{u}_{\varepsilon\delta}]) = 0$ .

Then, recalling (5.5) and (5.6), we conclude that

$$\begin{aligned} \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-1} (|\mathbf{L}\mathbf{u}_n| - G[\mathbf{u}_{\varepsilon\delta}]) &\leq \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-1} (G_n - G[\mathbf{u}_{\varepsilon\delta}]) \\ &\leq \left( \int_{Q_T} k_\varepsilon \right)^{\frac{1}{p}} \left( \int_{Q_T} k_\varepsilon |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^p \right)^{\frac{1}{p'}} \|G_n - G[\mathbf{u}_{\varepsilon\delta}]\|_{L^\infty(Q_T)} \\ &\leq C \|G_n - G[\mathbf{u}_{\varepsilon\delta}]\|_{L^\infty(Q_T)}. \end{aligned} \quad (5.13)$$

Going back to (5.11) and using (5.12) and the estimate above, we get

$$\langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle \leq \int_0^T \langle \partial_t \mathbf{u}_n, \mathbf{u}_n - \mathbf{u}_{\varepsilon\delta} \rangle_p + \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_n - \mathbf{u} \rangle + C \|G_n - G[\mathbf{u}_{\varepsilon\delta}]\|_{L^\infty(Q_T)} - \int_{Q_T} \mathbf{f} \cdot (\mathbf{u}_n - \mathbf{u}_{\varepsilon\delta}).$$

So, noticing that  $G[\mathbf{u}_{\varepsilon\delta}] \xrightarrow{\varepsilon \rightarrow 0} G[\mathbf{u}_\delta]$  in  $\mathcal{C}([0, T]; L^\infty(\Omega))$  and recalling (5.8), we obtain

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle &\leq \int_0^T \langle \partial_t \mathbf{u}_n, \mathbf{u}_n - \mathbf{u}_\delta \rangle_p + \int_0^T \langle \chi_\delta + \delta \Lambda_\delta, L(\mathbf{u}_n - \mathbf{u}) \rangle + \int_0^T \langle \mathbf{Y}_\delta, \mathbf{u}_n - \mathbf{u} \rangle \\ &\quad + C \|G_n - G[\mathbf{u}_\delta]\|_{L^\infty(Q_T)} - \int_{Q_T} \mathbf{f} \cdot (\mathbf{u}_n - \mathbf{u}_\delta). \end{aligned}$$

Step 2: The limit when  $\delta \rightarrow 0$ . Because there exists a positive constant  $C$  independent of  $\varepsilon$  and  $\delta$  such that

$$\begin{aligned}\|\mathcal{X}_\delta\|_{\mathcal{Y}'_p} &\leq \underline{\lim}_{\varepsilon \rightarrow 0} \|\mathbf{a}(L\mathbf{u}_{\varepsilon\delta})\|_{\mathcal{Y}'_p} \leq C, \\ \|\mathcal{Y}_\delta\|_{\mathcal{Y}'_p} &\leq \underline{\lim}_{\varepsilon \rightarrow 0} \|\mathbf{b}(\mathbf{u}_{\varepsilon\delta})\|_{L^{p'}(Q_T)^m} \leq C, \\ \|\Lambda_\delta\|_{L^{p'}(Q_T)^\ell} &\leq \underline{\lim}_{\varepsilon \rightarrow 0} \| |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-2} \mathbf{L}\mathbf{u}_{\varepsilon\delta} \|_{L^{p'}(Q_T)^\ell} \leq C,\end{aligned}$$

we obtain, for a subsequence,

$$\begin{aligned}\mathcal{X}_\delta &\xrightarrow{\delta \rightarrow 0} \mathcal{X} \quad \text{in } \mathcal{Y}'_p, \\ \mathcal{Y}_\delta &\xrightarrow{\delta \rightarrow 0} \mathcal{Y} \quad \text{in } \mathcal{Y}'_p, \\ \delta \Lambda_\delta &\xrightarrow{\delta \rightarrow 0} 0 \quad \text{in } L^{p'}(Q_T)^\ell.\end{aligned}$$

Since  $G[\mathbf{u}_\delta] \xrightarrow{\delta \rightarrow 0} G[\mathbf{u}]$  in  $L^\infty(Q_T)$ , we obtain

$$\begin{aligned}\overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle &\leq \int_0^T \langle \partial_t \mathbf{u}_n, \mathbf{u}_n - \mathbf{u} \rangle_p + \int_0^T \langle \mathcal{X}, L(\mathbf{u}_n - \mathbf{u}) \rangle + \int_0^T \langle \mathcal{Y}, \mathbf{u}_n - \mathbf{u} \rangle \\ &\quad + C \|G_n - G[\mathbf{u}]\|_{L^\infty(Q_T)} - \int_{Q_T} \mathbf{f} \cdot (\mathbf{u}_n - \mathbf{u}).\end{aligned}$$

Letting  $n \rightarrow \infty$  in the above inequality, using that  $\int_0^T \langle \partial_t \mathbf{u}_n, \mathbf{u}_n - \mathbf{u} \rangle_p \leq 0$  and  $G_n \xrightarrow{n} G[\mathbf{u}]$  in  $\mathcal{C}([0, T]; L^\infty(\Omega))$ , we conclude that

$$\overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle \leq 0.$$

If  $\mathcal{A} : \mathcal{Y}_p \rightarrow \mathcal{Y}'_p$  is defined by

$$\langle \mathcal{A}\mathbf{v}, \mathbf{w} \rangle = \int_{Q_T} \mathbf{a}(L\mathbf{v}) \cdot L\mathbf{w} + \int_{Q_T} \mathbf{b}(\mathbf{v}) \cdot \mathbf{w},$$

then

$$\langle \mathcal{A}\mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle = \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle - \delta \int_{Q_T} |\mathbf{L}\mathbf{u}_{\varepsilon\delta}|^{p-2} \mathbf{L}\mathbf{u}_{\varepsilon\delta} \cdot L(\mathbf{u}_{\varepsilon\delta} - \mathbf{u})$$

yields

$$\overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \langle \mathcal{A}\mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle = \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \langle \mathcal{A}_\delta \mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle \leq 0.$$

Step 3: Conclusion. Let  $a_\delta = \overline{\lim}_{\varepsilon \rightarrow 0} \langle \mathcal{A}\mathbf{u}_{\varepsilon\delta}, \mathbf{u}_{\varepsilon\delta} - \mathbf{u} \rangle$ . As  $\overline{\lim}_{\delta \rightarrow 0} a_\delta \leq 0$ , given  $\eta > 0$ , there exists  $\delta_\eta > 0$  such that  $a_{\delta_\eta} < \frac{\eta}{2}$ . But, because

$$\overline{\lim}_{\varepsilon \rightarrow 0} \langle \mathcal{A}\mathbf{u}_{\varepsilon\delta_\eta}, \mathbf{u}_{\varepsilon\delta_\eta} - \mathbf{u} \rangle = a_{\delta_\eta},$$

we can find  $\varepsilon_\eta = \varepsilon(\delta_\eta) > 0$  for which  $\langle \mathcal{A}\mathbf{u}_{\varepsilon_\eta\delta_\eta}, \mathbf{u}_{\varepsilon_\eta\delta_\eta} - \mathbf{u} \rangle < \eta$ . So,

$$\overline{\lim}_{\eta \rightarrow 0} \langle \mathcal{A}\mathbf{u}_{\varepsilon_\eta\delta_\eta}, \mathbf{u}_{\varepsilon_\eta\delta_\eta} - \mathbf{u} \rangle \leq 0.$$

From now on we set  $\mathbf{u}_\eta = \mathbf{u}_{\varepsilon_\eta\delta_\eta}$  and  $k_\eta = k_{\varepsilon_\eta}$ .

As the operator  $\mathcal{A}$  is bounded, monotone and hemicontinuous, it is pseudo-monotone. Therefore, as  $\overline{\lim}_{\eta \rightarrow 0} \langle \mathcal{A}\mathbf{u}_\eta, \mathbf{u}_\eta - \mathbf{u} \rangle \leq 0$ , we obtain

$$\langle \mathcal{A}\mathbf{u}, \mathbf{u} - \mathbf{v} \rangle \leq \underline{\lim}_{\eta \rightarrow 0} \langle \mathcal{A}\mathbf{u}_\eta, \mathbf{u}_\eta - \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in \mathbb{K}_{G[\mathbf{u}]}.$$

Finally, we conclude, going back to (5.9), that if  $\mathbf{v} \in \mathbb{K}_{G[\mathbf{u}]}$ , then

$$\langle \mathcal{A}\mathbf{u}, \mathbf{u} - \mathbf{v} \rangle \leq \underline{\lim}_{\eta \rightarrow 0} \left( \frac{1}{2} \int_\Omega |\mathbf{u}_0 - \mathbf{v}(0)|^2 + \int_0^T \langle \partial_t \mathbf{v}, \mathbf{v} - \mathbf{u}_\eta \rangle_p + \int_{Q_T} k_\eta |\mathbf{L}\mathbf{u}_\eta|^{p-2} \mathbf{L}\mathbf{u}_\eta \cdot L(\mathbf{v} - \mathbf{u}_\eta) - \int_{Q_T} \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}_\eta) \right).$$

But, as  $\mathbf{v} \in \mathbb{K}_{G[\mathbf{u}]}$ , as in (5.12) and (5.13),

$$\begin{aligned} \int_{Q_T} k_\eta |\mathbf{L}\mathbf{u}_\eta|^{p-2} \mathbf{L}\mathbf{u}_\eta \cdot L(\mathbf{v} - \mathbf{u}_\eta) &\leq \int_{Q_T} k_\eta |\mathbf{L}\mathbf{u}_\eta|^{p-1} (|\mathbf{L}\mathbf{v}| - |\mathbf{L}\mathbf{u}_\eta|) \\ &\leq \int_{Q_T} k_\eta |\mathbf{L}\mathbf{u}_\eta|^{p-1} (G[\mathbf{u}_\eta] - |\mathbf{L}\mathbf{u}_\eta|) + \int_{Q_T} k_\eta |\mathbf{L}\mathbf{u}_\eta|^{p-1} (G[\mathbf{u}] - G[\mathbf{u}_\eta]) \\ &\leq C \|G[\mathbf{u}] - G[\mathbf{u}_\eta]\|_{\mathcal{C}([0,T];L^\infty(\Omega))}. \end{aligned}$$

Then

$$\lim_{\eta \rightarrow 0} \int_{Q_T} k_\eta |\mathbf{L}\mathbf{u}_\eta|^{p-2} \mathbf{L}\mathbf{u}_\eta \cdot L(\mathbf{v} - \mathbf{u}_\eta) \leq 0$$

because  $\lim_{\eta \rightarrow 0} G[\mathbf{u}_\eta] = G[\mathbf{u}]$  in  $\mathcal{C}([0, T]; L^\infty(\Omega))$ . Hence,

$$\langle \mathcal{A}\mathbf{u}, \mathbf{u} - \mathbf{v} \rangle \leq \frac{1}{2} \int_{\Omega} |\mathbf{u}_0 - \mathbf{v}(0)|^2 + \int_0^T \langle \partial_t \mathbf{v}, \mathbf{v} - \mathbf{u} \rangle - \int_{Q_T} \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}),$$

concluding the proof since we already know that  $\mathbf{u} \in \mathbb{K}_{G[\mathbf{u}]}$ .  $\square$

*Proof of Theorem 2.8.* Let  $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{K}_g$  be two solutions of (2.4) and denote by  $\{\mathbf{w}_n\}_n$  and by  $\{g_n\}_n$  the regularizing sequences of Lemma 5.3 of  $\mathbf{w} = \frac{\mathbf{u}_1 + \mathbf{u}_2}{2} \in \mathbb{K}_g$  and  $g$ , respectively, with  $\mathbf{z} = \mathbf{u}_0$ . Considering

$$\varepsilon_n = \|g_n - g\|_{\mathcal{C}([0,T];L^\infty(\Omega))} \quad \text{and} \quad \rho_n = \frac{g^*}{g^* + \varepsilon_n} \xrightarrow{n} 1,$$

we have  $\widehat{\mathbf{w}}_n = \rho_n \mathbf{w}_n \in \mathbb{K}_g \cap \mathcal{B}_p$  and it may be chosen as test function in (2.4) for  $\mathbf{u}_1$  and  $\mathbf{u}_2$ . We obtain, by addition,

$$2 \int_0^T \langle \partial_t \widehat{\mathbf{w}}_n, \mathbf{w} - \widehat{\mathbf{w}}_n \rangle_p + \langle \mathcal{A}\mathbf{u}_1, \mathbf{u}_1 - \widehat{\mathbf{w}}_n \rangle + \langle \mathcal{A}\mathbf{u}_2, \mathbf{u}_2 - \widehat{\mathbf{w}}_n \rangle \leq 2 \int_{Q_T} \mathbf{f} \cdot (\mathbf{w} - \widehat{\mathbf{w}}_n) + (\rho_n - 1)^2 \int_{\Omega} |\mathbf{u}_0|^2. \quad (5.14)$$

Observing that

$$\langle \partial_t \widehat{\mathbf{w}}_n, \widehat{\mathbf{w}}_n - \mathbf{w} \rangle_p = \rho_n \langle \partial_t \mathbf{w}_n, \mathbf{w}_n - \mathbf{w} \rangle_p + \rho_n (\rho_n - 1) \langle \partial_t \mathbf{w}_n, \mathbf{w}_n \rangle_p$$

and integrating in time, since  $\mathbf{w}_n \in W^{1,p}(0, T; \mathbb{X}_p) \subset \mathcal{C}(0, T; L^2(\Omega)^m)$ , we have

$$\begin{aligned} \int_0^T \langle \partial_t \widehat{\mathbf{w}}_n, \widehat{\mathbf{w}}_n - \mathbf{w} \rangle_p &= \rho_n \int_0^T \langle \partial_t \mathbf{w}_n, \mathbf{w}_n - \mathbf{w} \rangle_p + \frac{1}{2} \rho_n (1 - \rho_n) (\|\mathbf{u}_0\|_{L^2(\Omega)^m}^2 - \|\mathbf{w}_n(T)\|_{L^2(\Omega)^m}^2) \\ &\leq \frac{1}{2} \rho_n (1 - \rho_n) \|\mathbf{u}_0\|_{L^2(\Omega)^m}^2 \xrightarrow{n} 0. \end{aligned}$$

Therefore, taking the limit in (5.14), since  $\widehat{\mathbf{w}}_n \xrightarrow{n} \frac{\mathbf{u}_1 + \mathbf{u}_2}{2}$ , we obtain

$$\int_{Q_T} (\mathbf{a}(\mathbf{L}\mathbf{u}_1) - \mathbf{a}(\mathbf{L}\mathbf{u}_2)) \cdot (\mathbf{L}\mathbf{u}_1 - \mathbf{L}\mathbf{u}_2) + \int_{Q_T} (\mathbf{b}(\mathbf{u}_1) - \mathbf{b}(\mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) = \langle \mathcal{A}\mathbf{u}_1 - \mathcal{A}\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2 \rangle \leq 0,$$

and the conclusion  $\mathbf{u}_1 = \mathbf{u}_2$  follows by the strict monotonicity of  $\mathbf{b}$  or  $\mathbf{a}$  with Assumption 2.2.  $\square$

## 6 Solution of the variational inequality

We study now the variational inequality case as well as the continuous dependence of its solution on the given data. We obtain different stability results whether we consider the case where the operator  $\mathbf{a}$  is monotone or strongly monotone.

*Proof of Theorem 2.10.* We penalize the variational inequality using the function  $k_\varepsilon$  defined in (4.1), as we have done in Section 4. For  $\varepsilon \in (0, 1)$  and  $\delta > 0$ , let us consider the problem of finding

$$\mathbf{w}_{\varepsilon\delta} \in \mathcal{Y}_p \cap H^1(0, T; L^2(\Omega))$$

such that

$$\left\{ \begin{array}{l} \int_{\Omega} \partial_t \mathbf{w}_{\varepsilon\delta}(t) \cdot \boldsymbol{\psi} + \int_{\Omega} \mathbf{a}(t, L\mathbf{w}_{\varepsilon\delta}(t)) \cdot L\boldsymbol{\psi} + \int_{\Omega} \mathbf{b}(t, \mathbf{w}_{\varepsilon\delta}(t)) \cdot L\boldsymbol{\psi} \\ \quad + \int_{\Omega} (\delta + k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}(t)|^p - g(t)^p)) |L\mathbf{w}_{\varepsilon\delta}(t)|^{p-2} L\mathbf{w}_{\varepsilon\delta}(t) \cdot L\boldsymbol{\psi} \\ = \int_{\Omega} \mathbf{f}(t) \cdot \boldsymbol{\psi} \quad \text{for all } \boldsymbol{\psi} \in \mathbb{X}_p \text{ for a.e. } t \in (0, T), \\ \mathbf{w}_{\varepsilon\delta}(0) = \mathbf{w}_0. \end{array} \right. \quad (6.1)$$

The proof of the existence of a solution for this problem is similar to the proof of Proposition 4.2 and can be done with the Galerkin method (see, for instance, [33, p. 240]). We observe that we consider here the function  $k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}|^p - g^p)$  instead of  $k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}| - G[\mathbf{w}_{\varepsilon\delta}])$ .

As in estimates (4.5), (4.6) and (5.5), we obtain, with a constant  $C > 0$  independent of  $\varepsilon$  and  $\delta$ ,

$$\|\mathbf{w}_{\varepsilon\delta}\|_{L^\infty(0, T; L^2(\Omega)^m)} \leq C, \quad (6.2)$$

$$\|L\mathbf{w}_{\varepsilon\delta}\|_{L^p(Q_T)^\ell} \leq \frac{C}{\delta^{1/p}}, \quad (6.3)$$

$$\|k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}|^p - g^p)\|_{L^1(Q_T)} \leq C. \quad (6.4)$$

Using Galerkin's approximation, we can argue formally with  $\partial_t \mathbf{w}_{\varepsilon\delta}$  as a test function on (6.1) and we get

$$\begin{aligned} & \int_{\Omega} |\partial_t \mathbf{w}_{\varepsilon\delta}(t)|^2 + \int_{\Omega} \mathbf{a}(t, L\mathbf{w}_{\varepsilon\delta}(t)) \cdot \partial_t L\mathbf{w}_{\varepsilon\delta}(t) + \int_{\Omega} \mathbf{b}(t, \mathbf{w}_{\varepsilon\delta}(t)) \cdot \partial_t \mathbf{w}_{\varepsilon\delta}(t) \\ & \quad + \int_{\Omega} (\delta + k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}|^p - g(t)^p)) |L\mathbf{w}_{\varepsilon\delta}(t)|^{p-2} L\mathbf{w}_{\varepsilon\delta}(t) \cdot \partial_t L\mathbf{w}_{\varepsilon\delta}(t) \\ & = \int_{\Omega} \mathbf{f}(t) \cdot \partial_t \mathbf{w}_{\varepsilon\delta}(t). \end{aligned} \quad (6.5)$$

Set  $\phi_\varepsilon(s) = \int_0^s k_\varepsilon(\tau) d\tau$  and observe that

$$\begin{aligned} & k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}(t)|^p - g^p(t)) |L\mathbf{w}_{\varepsilon\delta}(t)|^{p-2} L\mathbf{w}_{\varepsilon\delta}(t) \cdot \partial_t L\mathbf{w}_{\varepsilon\delta}(t) \\ & = \frac{1}{p} k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}(t)|^p - g^p(t)) \partial_t (|L\mathbf{w}_{\varepsilon\delta}(t)|^p - g^p(t)) + k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}(t)|^p - g^p(t)) g^{p-1}(t) \partial_t g(t) \\ & = \frac{1}{p} \partial_t (\phi_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}(t)|^p - g^p(t)) + k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}(t)|^p - g^p(t)) g^{p-1}(t) \partial_t g(t)). \end{aligned}$$

Integrating (6.5) between 0 and  $T$ , we obtain

$$\begin{aligned} & \int_{Q_T} |\partial_t \mathbf{w}_{\varepsilon\delta}|^2 + \int_{\Omega} A(T, L\mathbf{w}_{\varepsilon\delta}(T)) - \int_{\Omega} A(0, L\mathbf{w}_0) - \int_{Q_T} (\partial_t A)(L\mathbf{w}_{\varepsilon\delta}) + \frac{\delta}{p} \int_{\Omega} |L\mathbf{w}_{\varepsilon\delta}(T)|^p - \frac{\delta}{p} \int_{\Omega} |L\mathbf{w}_0|^p \\ & \quad + \frac{1}{p} \int_{\Omega} \phi_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}(T)|^p - g^p(T)) - \frac{1}{p} \int_{\Omega} \phi_\varepsilon(|L\mathbf{w}_0|^p - g^p(0)) + \int_{Q_T} k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}|^p - g^p) g^{p-1} \partial_t g \\ & = \int_{Q_T} (\mathbf{f} - \mathbf{b}(\mathbf{w}_{\varepsilon\delta})) \cdot \partial_t \mathbf{w}_{\varepsilon\delta} \end{aligned} \quad (6.6)$$

since  $A$  satisfies (2.7). But

$$\phi_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}(T)|^p - g^p(T)) \geq 0, \quad \phi_\varepsilon(|L\mathbf{w}_0|^p - g^p(0)) = 0 \quad \text{because } |L\mathbf{w}_0| \leq g(0),$$

and, using assumption (2.9), the Hölder and Young inequalities,

$$\int_{Q_T} (\mathbf{f} - \mathbf{b}(\mathbf{w}_{\varepsilon\delta})) \cdot \partial_t \mathbf{w}_{\varepsilon\delta} \leq \int_{Q_T} (|\mathbf{f}| + \mathbf{b}^* |\mathbf{w}_{\varepsilon\delta}|) |\partial_t \mathbf{w}_{\varepsilon\delta}| \leq C(\|\mathbf{f}\|_{L^2(Q_T)^m}^2 + \mathbf{b}^* \|\mathbf{w}_{\varepsilon\delta}\|_{L^2(Q_T)^m}^2) + \frac{1}{2} \|\partial_t \mathbf{w}_{\varepsilon\delta}\|_{L^2(Q_T)^m}^2$$

and (2.8), from (6.6) we have

$$\begin{aligned} \frac{1}{2} \|\partial_t \mathbf{w}_{\varepsilon\delta}\|_{L^2(Q_T)^m}^2 &\leq C(\|\mathbf{f}\|_{L^2(Q_T)^m}^2 + \mathbf{b}^* \|\mathbf{w}_{\varepsilon\delta}\|_{L^2(Q_T)^m}^2) + 2\mathbf{a}^* \|\mathbf{L}\mathbf{w}_0\|_{L^p(\Omega)^{\ell}}^p + A_1 |Q_T| + A_2 \|\mathbf{L}\mathbf{w}_{\varepsilon\delta}\|_{L^p(Q_T)^{\ell}}^p \\ &\quad + \frac{\delta}{p} \|\mathbf{L}\mathbf{w}_0\|_{L^p(\Omega)^{\ell}}^p + \|k_{\varepsilon}(|\mathbf{L}\mathbf{w}_{\varepsilon\delta}|^p - g^p)\|_{L^1(Q_T)} \|\mathbf{g}\|_{L^p(Q_T)}^{p-1} \|\partial_t \mathbf{g}\|_{L^\infty(Q_T)}. \end{aligned}$$

From (6.2)–(6.4) we obtain, with a constant  $C' > 0$  independent of  $\varepsilon$  and  $\delta$ ,

$$\|\partial_t \mathbf{w}_{\varepsilon\delta}\|_{L^2(Q_T)^m} \leq C'(1 + \|\mathbf{L}\mathbf{w}_{\varepsilon\delta}\|_{L^p(Q_T)^{\ell}}^p). \quad (6.7)$$

Then, recalling that Assumption 2.5 implies, by the Aubin–Lions lemma, the compactness of  $\mathcal{D}_p \hookrightarrow \mathcal{H}$ , there exists a subsequence that we still denote by  $\{\mathbf{w}_{\varepsilon\delta}\}_{\varepsilon}$  such that, for every  $t \in (0, T]$ ,

$$\begin{aligned} \mathbf{w}_{\varepsilon\delta} &\xrightarrow{\varepsilon \rightarrow 0} \mathbf{w}_{\delta} \quad \text{in } L^2(Q_T)^m, \\ \mathbf{L}\mathbf{w}_{\varepsilon\delta} &\xrightarrow{\varepsilon \rightarrow 0} \mathbf{L}\mathbf{w}_{\delta} \quad \text{in } L^p(Q_T)^{\ell} \text{ weak}, \\ \partial_t \mathbf{w}_{\varepsilon\delta} &\xrightarrow{\varepsilon \rightarrow 0} \partial_t \mathbf{w}_{\delta} \quad \text{in } L^2(Q_T)^m \text{ weak}. \end{aligned}$$

Recalling Lemma 4.1 and observing that  $k_{\varepsilon}(|\mathbf{L}\mathbf{v}|^p - g^p) = 0$  if  $\mathbf{v} \in \mathbb{K}_g$ , we have, for any  $t \in (0, T]$ ,

$$\begin{aligned} &\int_{Q_t} k_{\varepsilon}(|\mathbf{L}\mathbf{w}_{\varepsilon\delta}|^p - g^p) |\mathbf{L}\mathbf{w}_{\varepsilon\delta}|^{p-2} \mathbf{L}\mathbf{w}_{\varepsilon\delta} \cdot \mathbf{L}(\mathbf{v} - \mathbf{w}_{\varepsilon\delta}) \\ &= \int_{Q_t} (k_{\varepsilon}(|\mathbf{L}\mathbf{w}_{\varepsilon\delta}|^p - g^p) |\mathbf{L}\mathbf{w}_{\varepsilon\delta}|^{p-2} \mathbf{L}\mathbf{w}_{\varepsilon\delta} - k_{\varepsilon}(|\mathbf{L}\mathbf{v}|^p - g^p) |\mathbf{L}\mathbf{v}|^{p-2} \mathbf{L}\mathbf{v}) \cdot \mathbf{L}(\mathbf{v} - \mathbf{w}_{\varepsilon\delta}) \\ &\quad + \int_{Q_t} k_{\varepsilon}(|\mathbf{L}\mathbf{v}|^p - g^p) |\mathbf{L}\mathbf{v}|^{p-2} \mathbf{L}\mathbf{v} \cdot \mathbf{L}(\mathbf{v} - \mathbf{w}_{\varepsilon\delta}) \leq 0 \quad \text{for all } \mathbf{v} \in \mathbb{K}_g. \end{aligned} \quad (6.8)$$

Using  $\mathbf{v} - \mathbf{w}_{\varepsilon\delta}$  as test function in (6.1) and integrating over  $(0, t)$ , by (6.8) and by the monotonicity of the operators  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\xi \mapsto |\mathbf{L}\xi|^{p-2} \mathbf{L}\xi$ , we obtain

$$\int_{Q_t} \partial_t \mathbf{w}_{\varepsilon\delta} \cdot (\mathbf{v} - \mathbf{w}_{\varepsilon\delta}) + \int_{Q_t} \mathbf{a}(\mathbf{L}\mathbf{v}) \cdot \mathbf{L}(\mathbf{v} - \mathbf{w}_{\varepsilon\delta}) + \int_{Q_t} \mathbf{b}(\mathbf{v}) \cdot (\mathbf{v} - \mathbf{w}_{\varepsilon\delta}) + \delta \int_{Q_t} |\mathbf{L}\mathbf{v}|^{p-2} \mathbf{L}\mathbf{v} \cdot \mathbf{L}(\mathbf{v} - \mathbf{w}_{\varepsilon\delta}) \geq \int_{Q_t} \mathbf{f} \cdot (\mathbf{v} - \mathbf{w}_{\varepsilon\delta}).$$

Passing to the limit when  $\varepsilon$  tends to zero, we get

$$\begin{aligned} &\int_{Q_t} \partial_t \mathbf{w}_{\delta} \cdot (\mathbf{v} - \mathbf{w}_{\delta}) + \int_{Q_t} \mathbf{a}(\mathbf{L}\mathbf{v}) \cdot \mathbf{L}(\mathbf{v} - \mathbf{w}_{\delta}) + \int_{Q_t} \mathbf{b}(\mathbf{v}) \cdot (\mathbf{v} - \mathbf{w}_{\delta}) + \delta \int_{Q_t} |\mathbf{L}\mathbf{v}|^{p-2} \mathbf{L}\mathbf{v} \cdot \mathbf{L}(\mathbf{v} - \mathbf{w}_{\delta}) \\ &\geq \int_{Q_t} \mathbf{f} \cdot (\mathbf{v} - \mathbf{w}_{\delta}). \end{aligned} \quad (6.9)$$

Arguing as in Lemma 5.2, we also prove that  $\mathbf{w}_{\delta} \in \mathbb{K}_g$ .

The next step is to let  $\delta \rightarrow 0$ . From (6.7) we have

$$\|\partial_t \mathbf{w}_{\delta}\|_{L^2(Q_T)^m} \leq \liminf_{\varepsilon \rightarrow 0} \|\partial_t \mathbf{w}_{\varepsilon\delta}\|_{L^2(Q_T)^m} \leq C' \left( 1 + \liminf_{\varepsilon \rightarrow 0} \int_{Q_T} |\mathbf{L}(\mathbf{w}_{\varepsilon\delta})|^p \right).$$

Using the sets defined in (5.7), we get

$$\begin{aligned} \int_{Q_T} |\mathbf{L}\mathbf{w}_{\varepsilon\delta}|^p &= \int_{A_{\varepsilon\delta}} |\mathbf{L}\mathbf{w}_{\varepsilon\delta}|^p + \int_{B_{\varepsilon\delta}} (|\mathbf{L}\mathbf{w}_{\varepsilon\delta}|^p - g^p + g^p) + \int_{C_{\varepsilon\delta}} |\mathbf{L}\mathbf{w}_{\varepsilon\delta}|^p \\ &\leq 2\|\mathbf{g}\|_{L^p(Q_T)}^p + |Q_T| + \int_{B_{\varepsilon\delta}} \varepsilon k_{\varepsilon}(|\mathbf{L}\mathbf{w}_{\varepsilon\delta}|^p - g^p) + \frac{C_1}{e^{1/\varepsilon^2} - 1} \leq C \end{aligned}$$

because, for  $s \in (0, \frac{1}{\varepsilon})$ ,  $\frac{s}{\varepsilon} \leq k_\varepsilon(s)$ , and by (5.6),

$$\int_{C_{\varepsilon\delta}} (e^{1/\varepsilon^2} - 1) |L\mathbf{w}_{\varepsilon\delta}|^p = \int_{C_{\varepsilon\delta}} k_\varepsilon(|L\mathbf{w}_{\varepsilon\delta}|^p - g^p) |L\mathbf{w}_{\varepsilon\delta}|^p \leq C_1,$$

so  $\{\partial_t \mathbf{w}_\delta\}_\delta$  is also uniformly bounded in  $L^2(Q_T)^m$ . Since  $\mathbf{w}_\delta \in \mathbb{K}_g$ , we have  $|L\mathbf{w}_\delta|$  bounded in  $L^\infty(Q_T)$  independently of  $\delta$ . Then, for a subsequence, we have

$$\begin{aligned} \mathbf{w}_\delta &\xrightarrow{\delta \rightarrow 0} \mathbf{w} && \text{in } H^1(0, T; L^2(Q_T)^m) \text{ weak,} \\ L\mathbf{w}_\delta &\xrightarrow{\delta \rightarrow 0} L\mathbf{w} && \text{in } L^\infty(Q_T)^\ell \text{ weak-*,} \\ \mathbf{w}_\delta(t) &\xrightarrow{\delta \rightarrow 0} \mathbf{w}(t) && \text{in } L^2(\Omega)^m \text{ weak for all } 0 < t \leq T. \end{aligned}$$

We can pass to the limit when  $\delta \rightarrow 0$  in (6.9), writing

$$\begin{aligned} &\int_{Q_t} \partial_t \mathbf{w}_\delta \cdot \mathbf{v} + \int_{Q_t} \mathbf{a}(L\mathbf{v}) \cdot L(\mathbf{v} - \mathbf{w}_\delta) + \int_{Q_t} \mathbf{b}(\mathbf{v}) \cdot (\mathbf{v} - \mathbf{w}_\delta) + \delta \int_{Q_t} |L\mathbf{v}|^{p-2} L\mathbf{v} \cdot L(\mathbf{v} - \mathbf{w}_\delta) \\ &\geq \int_{Q_t} \mathbf{f} \cdot (\mathbf{v} - \mathbf{w}_\delta) + \frac{1}{2} \int_{\Omega} |\mathbf{w}_\delta(t)|^2 - \frac{1}{2} \int_{\Omega} |\mathbf{w}_0|^2. \end{aligned}$$

Because  $\mathbf{w}_\delta(t) \rightharpoonup \mathbf{w}(t)$  in  $L^2(\Omega)^m$ -weak yields

$$\liminf_{\delta \rightarrow 0} \int_{\Omega} |\mathbf{w}_\delta(t)|^2 \geq \int_{\Omega} |\mathbf{w}(t)|^2$$

for each  $0 < t \leq T$ , we recover in the limit that  $\mathbf{w}$  satisfies

$$\int_{Q_t} \partial_t \mathbf{w} \cdot (\mathbf{v} - \mathbf{w}) + \int_{Q_t} \mathbf{a}(L\mathbf{v}) \cdot L(\mathbf{v} - \mathbf{w}) + \int_{Q_t} \mathbf{b}(\mathbf{v}) \cdot (\mathbf{v} - \mathbf{w}) \geq \int_{Q_t} \mathbf{f} \cdot (\mathbf{v} - \mathbf{w}) \quad \text{for all } \mathbf{v} \in \mathbb{K}_g.$$

Finally, as in the proof of Theorem 2.7,  $\mathbf{w}$  also belongs to  $\mathbb{K}_g$ . We may apply Minty's lemma and conclude that it solves (2.6).

The uniqueness of the solution is immediate since if  $\mathbf{w}_1$  and  $\mathbf{w}_2$  are two solutions of (2.6), then

$$\int_{Q_t} \partial_t (\mathbf{w}_1 - \mathbf{w}_2) \cdot (\mathbf{w}_1 - \mathbf{w}_2) + \int_{Q_t} (\mathbf{a}(L\mathbf{w}_1) - \mathbf{a}(L\mathbf{w}_2)) \cdot L(\mathbf{w}_1 - \mathbf{w}_2) + \int_{Q_t} (\mathbf{b}(\mathbf{w}_1) - \mathbf{b}(\mathbf{w}_2)) \cdot (\mathbf{w}_1 - \mathbf{w}_2) \leq 0$$

and, by monotonicity of  $\mathbf{a}$  and  $\mathbf{b}$ , we get

$$\int_{\Omega} |\mathbf{w}_1(t) - \mathbf{w}_2(t)|^2 \leq 0 \quad \text{for all } t \in (0, T),$$

concluding that  $\mathbf{w}_1 = \mathbf{w}_2$ . □

Next we prove the stability of the solutions of the variational inequality (2.6) with respect to the given data. The results we obtain depend on the assumptions on  $\mathbf{a}$ , and we are able to give a result even in the very degenerate case  $\mathbf{a} \equiv \mathbf{0}$  and  $\mathbf{b} \equiv \mathbf{0}$ .

*Proof of Theorem 2.11.* Considering the threshold functions  $g_1$  and  $g_2$  satisfying (2.10), let  $\mathbb{K}_{g_1(t)}$  and  $\mathbb{K}_{g_2(t)}$  be, respectively, the corresponding convex sets defined in (2.5). For  $\beta(t) = \|g_i(t) - g_j(t)\|_{L^\infty(\Omega)}$  for  $i, j \in \{1, 2\}$ ,  $i \neq j$ , and given  $\mathbf{w}_i$  such that  $\mathbf{w}_i(t) \in \mathbb{K}_{g_i(t)}$  for a.e.  $t \in (0, T)$ , we define the functions

$$\mathbf{w}_{ij}(t) = \frac{g_* \mathbf{w}_i(t)}{g_* + \beta(t)} \in \mathbb{K}_{g_j(t)}.$$

Choosing

$$C \geq \max \left\{ \frac{1}{g_*} \|\mathbf{w}_i\|_{L^2(Q_T)^m}, \frac{1}{g_*} \|L\mathbf{w}_i\|_{L^p(Q_T)^\ell} \right\},$$

we observe that

$$\|\mathbf{w}_i(t) - \mathbf{w}_{ij}(t)\|_{L^2(\Omega)^m} \leq C\beta(t) \quad \text{and} \quad \|L(\mathbf{w}_i(t) - \mathbf{w}_{ij}(t))\|_{L^p(\Omega)^\ell} \leq C\beta(t). \quad (6.10)$$

Considering, for  $i, j \in \{1, 2\}$ ,  $i \neq j$ , the solution  $\mathbf{w}_i$  of the variational inequality (2.6) associated to the constraint  $g_i$ , and using  $\mathbf{w}_{j_i}$  as test function, we have, for  $t \in (0, T]$ ,

$$\int_{Q_t} \partial_t \mathbf{w}_i \cdot (\mathbf{w}_i - \mathbf{w}_{j_i}) + \int_{Q_t} \mathbf{a}(L\mathbf{w}_i) \cdot L(\mathbf{w}_i - \mathbf{w}_{j_i}) + \int_{Q_t} \mathbf{b}(\mathbf{w}_i) \cdot (\mathbf{w}_i - \mathbf{w}_{j_i}) \leq \int_{Q_t} \mathbf{f}_i \cdot (\mathbf{w}_i - \mathbf{w}_{j_i}),$$

and so

$$\begin{aligned} & \int_{Q_t} \partial_t \mathbf{w}_i \cdot (\mathbf{w}_i - \mathbf{w}_j) + \int_{Q_t} \mathbf{a}(L\mathbf{w}_i) \cdot L(\mathbf{w}_i - \mathbf{w}_j) + \int_{Q_t} \mathbf{b}(\mathbf{w}_i) \cdot (\mathbf{w}_i - \mathbf{w}_j) \\ & \leq \int_{Q_t} \mathbf{f}_i \cdot (\mathbf{w}_i - \mathbf{w}_j) + \int_{Q_t} \partial_t \mathbf{w}_i \cdot (\mathbf{w}_{j_i} - \mathbf{w}_j) + \int_{Q_t} \mathbf{a}(L\mathbf{w}_i) \cdot L(\mathbf{w}_{j_i} - \mathbf{w}_j) \\ & \quad + \int_{Q_t} \mathbf{b}(\mathbf{w}_i) \cdot (\mathbf{w}_{j_i} - \mathbf{w}_j) + \int_{Q_t} \mathbf{f}_i \cdot (\mathbf{w}_j - \mathbf{w}_{j_i}). \end{aligned}$$

Adding the inequalities we obtained in the former expression to  $(i, j) = (1, 2)$  and  $(i, j) = (2, 1)$ , and setting  $\mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2$ , we get

$$\int_{Q_t} \partial_t \mathbf{w} \cdot \mathbf{w} + \int_{Q_t} (\mathbf{a}(L\mathbf{w}_1) - \mathbf{a}(L\mathbf{w}_2)) \cdot L\mathbf{w} + \int_{Q_t} (\mathbf{b}(\mathbf{w}_1) - \mathbf{b}(\mathbf{w}_2)) \cdot \mathbf{w} \leq \int_{Q_t} (\mathbf{f}_1 - \mathbf{f}_2) \cdot \mathbf{w} + \Theta(t), \quad (6.11)$$

where

$$\begin{aligned} \Theta(t) = & \int_{Q_t} \partial_t \mathbf{w}_1 \cdot (\mathbf{w}_{21} - \mathbf{w}_2) + \int_{Q_t} \partial_t \mathbf{w}_2 \cdot (\mathbf{w}_{12} - \mathbf{w}_1) + \int_{Q_t} \mathbf{a}(L\mathbf{w}_1) \cdot L(\mathbf{w}_{21} - \mathbf{w}_2) \\ & + \int_{Q_t} \mathbf{a}(L\mathbf{w}_2) \cdot L(\mathbf{w}_{12} - \mathbf{w}_1) + \int_{Q_t} \mathbf{b}(\mathbf{w}_1) \cdot (\mathbf{w}_{21} - \mathbf{w}_2) + \int_{Q_t} \mathbf{b}(\mathbf{w}_2) \cdot (\mathbf{w}_{12} - \mathbf{w}_1) \\ & + \int_{Q_t} \mathbf{f}_1 \cdot (\mathbf{w}_2 - \mathbf{w}_{21}) + \int_{Q_t} \mathbf{f}_2 \cdot (\mathbf{w}_1 - \mathbf{w}_{12}). \end{aligned}$$

Estimates (6.2), (6.3), (6.7) and (6.10) allow us to conclude that there exists a positive constant  $C$  such that, for any  $t \in (0, T)$ ,

$$\Theta(t) \leq C \int_0^t \|\mathbf{g}_1(\tau) - \mathbf{g}_2(\tau)\|_{L^\infty(\Omega)} d\tau. \quad (6.12)$$

From (6.11) and (6.12), using the monotonicity of  $\mathbf{a}$  and  $\mathbf{b}$ , we obtain

$$\frac{d}{dt} \|\mathbf{w}(t)\|_{L^2(\Omega)^m}^2 \leq \|\mathbf{w}(t)\|_{L^2(\Omega)^m}^2 + \|\mathbf{f}_1(t) - \mathbf{f}_2(t)\|_{L^2(\Omega)^m}^2 + 2C\|\mathbf{g}_1(t) - \mathbf{g}_2(t)\|_{L^\infty(\Omega)}.$$

Applying the Gronwall inequality, we obtain

$$\|\mathbf{w}_1(t) - \mathbf{w}_2(t)\|_{L^2(\Omega)^m}^2 \leq e^T \left( \int_0^T \|\mathbf{f}_1(t) - \mathbf{f}_2(t)\|_{L^2(\Omega)^m}^2 dt + \|\mathbf{w}_{1_0} - \mathbf{w}_{2_0}\|_{L^2(\Omega)^m}^2 + \int_0^T \|\mathbf{g}_1(t) - \mathbf{g}_2(t)\|_{L^\infty(\Omega)} dt \right),$$

concluding (2.11).

Consider now the case where  $\mathbf{a}$  is strongly monotone. When  $p \geq 2$ , after integration, from (6.11) and (6.12) we obtain

$$\begin{aligned} & \frac{1}{2} \|\mathbf{w}(t)\|_{L^2(\Omega)^m}^2 + a_* \int_0^t \|L\mathbf{w}(\tau)\|_{L^p(\Omega)^\ell}^p d\tau \\ & \leq \int_0^t \|\mathbf{f}_1(\tau) - \mathbf{f}_2(\tau)\|_{L^2(\Omega)^m} \|\mathbf{w}(\tau)\|_{L^2(\Omega)^m} d\tau + \frac{1}{2} \|\mathbf{w}(0)\|_{L^2(\Omega)^m}^2 + C\|\mathbf{g}_1 - \mathbf{g}_2\|_{L^1(0, T; L^\infty(\Omega))}, \end{aligned}$$

and so we conclude that there exists another positive constant  $C$  depending on  $T$  such that

$$\begin{aligned} & \| \mathbf{w}_1 - \mathbf{w}_2 \|_{L^\infty(0, T; L^2(\Omega)^m)}^2 + \| \mathbf{w}_1 - \mathbf{w}_2 \|_{\mathcal{V}_p(Q_T)}^p \\ & \leq C (\| \mathbf{f}_1 - \mathbf{f}_2 \|_{L^2(Q_T)^m}^2 + \| \mathbf{w}_{10} - \mathbf{w}_{20} \|_{L^2(\Omega)^m}^2 + \| \mathbf{g}_1 - \mathbf{g}_2 \|_{L^1(0, T; L^\infty(\Omega))}), \end{aligned} \quad (6.13)$$

obtaining (2.13) when  $p \geq 2$ .

If  $1 < p < 2$ , using the strong monotonicity (2.12) of the operator  $\mathbf{a}$ , from (6.11) we get

$$\int_{Q_t} \partial_t \mathbf{w} \cdot \mathbf{w} + a_* \int_{Q_t} |L\mathbf{w}|^2 (|L\mathbf{w}_1| + |L\mathbf{w}_2|)^{p-2} \leq \int_{Q_t} (\mathbf{f}_1 - \mathbf{f}_2) \cdot \mathbf{w} + \Theta(t).$$

Applying the reverse Hölder inequality, we obtain

$$\int_{Q_t} \partial_t \mathbf{w} \cdot \mathbf{w} + a_* \left( \int_{Q_t} |L\mathbf{w}|^p \right)^{\frac{2}{p}} \left( \int_{Q_t} (|L\mathbf{w}_1| + |L\mathbf{w}_2|)^p \right)^{\frac{p-2}{2}} \leq \int_{Q_t} (\mathbf{f}_1 - \mathbf{f}_2) \cdot \mathbf{w} + \Theta(t). \quad (6.14)$$

Since  $p < 2$  and  $|L\mathbf{w}_i| \leq g^*$  a.e. in  $Q_T$ ,  $i = 1, 2$ , we have

$$\left( \int_{Q_t} (|L\mathbf{w}_1| + |L\mathbf{w}_2|)^p \right)^{\frac{p-2}{2}} \geq C,$$

where  $C$  is a positive constant depending only on  $\| \mathbf{f}_i \|_{L^{p'}(Q_T)^m}$  and  $\| \mathbf{w}_{i0} \|_{L^2(\Omega)^m}$ . From (6.14) we obtain

$$\frac{1}{2} \| \mathbf{w}(t) \|_{L^2(\Omega)^m}^2 + C a_* \left( \int_{Q_t} |L\mathbf{w}|^p \right)^{\frac{2}{p}} \leq \int_0^t \| \mathbf{f}_1 - \mathbf{f}_2 \|_{L^2(\Omega)^m} \| \mathbf{w} \|_{L^2(\Omega)^m} + \frac{1}{2} \| \mathbf{w}(0) \|_{L^2(\Omega)^m}^2 + \int_0^t \Theta,$$

and so, using the Hölder and Young inequalities, we get the inequality

$$\| \mathbf{w} \|_{L^\infty(0, T; L^2(\Omega)^m)}^2 + \| \mathbf{w} \|_{\mathcal{V}_p}^2 \leq C (\| \mathbf{f}_1 - \mathbf{f}_2 \|_{L^2(Q_T)^m}^2 + \| \mathbf{w}_{10} - \mathbf{w}_{20} \|_{L^2(\Omega)^m}^2 + \| \mathbf{g}_1 - \mathbf{g}_2 \|_{L^1(0, T; L^\infty(\Omega))}). \quad (6.15)$$

From (6.13) and (6.15) the conclusion follows.  $\square$

*Proof of Theorem 2.13.* Consider a sequence of solutions  $\mathbf{w}_n$  given by Theorem 2.10 for a sequence of  $g_n \in W^{1, \infty}(0, T; L^\infty(\Omega))$  such that

$$g_n \xrightarrow{n} g \quad \text{in } \mathcal{C}([0, T]; L^\infty(\Omega)).$$

First we show that  $\{ \mathbf{w}_n \}_n$  is relatively compact in  $\mathcal{C}([0, T]; L^2(\Omega))$ . For arbitrary  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\varepsilon_n = \sup_{|\tau-s|<\delta} \| g_n(\tau) - g_n(s) \|_{L^\infty(\Omega)} < \varepsilon$$

for all  $n$  sufficiently large and all  $\tau, s \in (0, T)$ . Setting

$$\rho_n = \frac{g^*}{g^* + \varepsilon_n},$$

we obtain  $\rho_n \mathbf{w}_n(s) \in \mathbb{K}_{g_n(\tau)}$  for all  $\tau \in (s - \delta, s + \delta)$ , and  $\rho_n \mathbf{w}_n(s)$  can be chosen as test function in (2.14) for  $\mathbf{w}_n$  at  $t = \tau$ , obtaining

$$\begin{aligned} & \int_{\Omega} \partial_t \mathbf{w}_n(\tau) \cdot (\rho_n \mathbf{w}_n(s) - \mathbf{w}_n(\tau)) + \int_{\Omega} \mathbf{a}(\tau, \mathbf{w}_n(\tau)) \cdot L(\rho_n \mathbf{w}_n(s) - \mathbf{w}_n(\tau)) + \int_{\Omega} \mathbf{b}(\tau, \mathbf{w}_n(\tau)) \cdot (\rho_n \mathbf{w}_n(s) - \mathbf{w}_n(\tau)) \\ & \geq \int_{\Omega} \mathbf{f}(\tau) \cdot (\rho_n \mathbf{w}_n(s) - \mathbf{w}_n(\tau)). \end{aligned} \quad (6.16)$$

Since  $0 < \rho_n \leq 1$  and the solutions  $\mathbf{w}_n \in \mathbb{K}_{g_n}$  are uniformly bounded in  $L^\infty(0, T; \mathcal{V}_p \cap L^2(\Omega)^m)$ , from (6.16) for fixed  $s$  we can integrate in  $\tau$  on  $[s, t]$  obtaining

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\mathbf{w}_n(t) - \mathbf{w}_n(s)|^2 &= \frac{1}{2} \int_s^t \frac{d}{d\tau} \int_{\Omega} |\mathbf{w}_n(\tau) - \mathbf{w}_n(s)|^2 \\ &\leq (\rho_n - 1) \int_s^t \int_{\Omega} \partial_\tau \mathbf{w}_n(\tau) \cdot \mathbf{w}_n(s) + C_1 |t - s| + C \int_s^t \|\mathbf{f}(\tau)\|_{L^2(\Omega)^m} \\ &\leq (\rho_n - 1) \int_{\Omega} (\mathbf{w}_n(t) - \mathbf{w}_n(s)) \cdot \mathbf{w}_n(s) + C' |t - s|^{\frac{1}{2}} (T^{\frac{1}{2}} + \|\mathbf{f}\|_{L^2(Q_T)^m}) \\ &\leq C'' (\varepsilon + |t - s|^{\frac{1}{2}}) \quad \text{for all } |t - s| < \delta \end{aligned}$$

and all  $n$  sufficiently large. Hence  $\{\mathbf{w}_n\}_n$  is equicontinuous on  $[0, T]$  with values in  $L^2(\Omega)^m$ . Therefore, we can take for a subsequence

$$\mathbf{w}_n \xrightarrow{n} \mathbf{w} \quad \text{in } \mathcal{C}([0, T]; L^2(\Omega)^m) \quad \text{and} \quad \mathbf{w}_n \xrightarrow{n} \mathbf{w} \quad \text{in } L^\infty(0, T; \mathbf{V}_p) \text{ weak-}^*$$

for some  $\mathbf{w}$  which is such that  $\mathbf{w} \in \mathbb{K}_g \cap \mathcal{V}_p \cap \mathcal{C}([0, T]; L^2(\Omega)^m)$  and  $\mathbf{w}(0) = \mathbf{w}_0$ .

We conclude that  $\mathbf{w}$  is a weak solution to (2.4) with  $\mathbb{K}_g$ . Using Minty's lemma and taking  $\mathbf{v}_n = \rho_n \mathbf{v} \in \mathbb{K}_{g_n}$  for arbitrary  $\mathbf{v} \in \mathbb{K}_g$  in

$$\int_0^T \langle \partial_t \mathbf{v}_n, \mathbf{v}_n - \mathbf{w}_n \rangle_p + \int_{Q_T} \mathbf{a}(L\mathbf{v}_n) \cdot L(\mathbf{v}_n - \mathbf{w}_n) + \int_{Q_T} \mathbf{b}(\mathbf{v}_n) \cdot (\mathbf{v}_n - \mathbf{w}_n) \geq \int_{Q_T} \mathbf{f} \cdot (\mathbf{v}_n - \mathbf{w}_n) - \frac{1}{2} \int_{\Omega} |\mathbf{w}_0 - \mathbf{v}_n(0)|^2,$$

we observe that  $\mathbf{v}_n \xrightarrow{n} \mathbf{v}$  in  $\mathcal{B}_p$ . By the uniqueness, the whole sequence  $\{\mathbf{w}_n\}_n$  converges to  $\mathbf{w}$ .  $\square$

## References

- [1] C. Amrouche and N. E. H. Seloula,  $L^p$ -theory for vector potentials and Sobolev's inequalities for vector fields: Application to the Stokes equations with pressure boundary conditions, *Math. Models Methods Appl. Sci.* **23** (2013), no. 1, 37–92.
- [2] A. Azevedo and L. Santos, A diffusion problem with gradient constraint depending on the temperature, *Adv. Math. Sci. Appl.* **20** (2010), no. 1, 153–168.
- [3] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Applications to free Boundary Problems*, John Wiley & Sons, New York, 1984.
- [4] J. W. Barrett and E. Süli, Reflections on Dubinskiĭ's nonlinear compact embedding theorem, *Publ. Inst. Math. (Beograd) (N.S.)* **91(105)** (2012), 95–110.
- [5] A. Bensoussan and J.-L. Lions, *Contrôle impulsionnel et inéquations quasi variationnelles*, Math. Models Methods Inform. Sci. 11, Gauthier-Villars, Paris, 1982.
- [6] M. Biroli, Sur les inéquations paraboliques avec convexe dépendant du temps: solution forte et solution faible, *Riv. Mat. Univ. Parma (3)* **3** (1974), 33–72.
- [7] H. Brezis, Équations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier (Grenoble)* **18** (1968), no. 1, 115–175.
- [8] L. Capogna, D. Danielli and N. Garofalo, Subelliptic mollifiers and a characterization of Rellich and Poincaré domains, *Rend. Semin. Mat. Univ. Politec. Torino* **51** (1993), no. 4, 361–386.
- [9] D. Danielli, A compact embedding theorem for a class of degenerate Sobolev spaces, *Rend. Semin. Mat. Univ. Politec. Torino* **49** (1991), no. 3, 399–420.
- [10] M. Derridj and J. A.-P. Dias, Le problème de Dirichlet pour une classe d'opérateurs non linéaires, *J. Math. Pures Appl. (9)* **51** (1972), 219–230.
- [11] T. Fukao and N. Kenmochi, Parabolic variational inequalities with weakly time-dependent constraints, *Adv. Math. Sci. Appl.* **23** (2013), no. 2, 365–395.
- [12] P. Hajtász and P. Koskela, Sobolev met Poincaré, *Mem. Amer. Math. Soc.* **145** (2000), no. 688, 1–101.
- [13] M. Hintermüller and C. N. Rautenberg, Parabolic quasi-variational inequalities with gradient-type constraints, *SIAM J. Optim.* **23** (2013), no. 4, 2090–2123.

- [14] M. Hintermüller and C. N. Rautenberg, On the uniqueness and numerical approximation of solutions to certain parabolic quasi-variational inequalities, *Port. Math.* **74** (2017), no. 1, 1–35.
- [15] M. Hintermüller, C. N. Rautenberg and N. Strogies, Dissipative and non-dissipative evolutionary quasi-variational inequalities with gradient constraints, *Set-Valued Var. Anal.* (2018), DOI 10.1007/s11228-018-0489-0.
- [16] R. Kano, Y. Murase and N. Kenmochi, Nonlinear evolution equations generated by subdifferentials with nonlocal constraints, in: *Nonlocal and Abstract Parabolic Equations and Their Applications*, Banach Center Publ. 86, Polish Academy of Sciences, Warsaw (2009), 175–194.
- [17] N. Kenmochi, Solvability of nonlinear evolution equations with time-dependent constraints and applications, *Bull. Fac. Educ. Chiba Univ. Part II* **30** (1981), 1–87.
- [18] N. Kenmochi, Parabolic quasi-variational diffusion problems with gradient constraints, *Discrete Contin. Dyn. Syst. Ser. S* **6** (2013), no. 2, 423–438.
- [19] N. Kenmochi and M. Niezgodka, Weak solvability for parabolic variational inclusions and application to quasi-variational problems, *Adv. Math. Sci. Appl.* **25** (2016), no. 1, 63–98.
- [20] M. Kubo, Quasi-variational analysis, *Sugaku Expositions* **30** (2017), no. 1, 17–34.
- [21] M. Kunze and J. F. Rodrigues, An elliptic quasi-variational inequality with gradient constraints and some of its applications, *Math. Methods Appl. Sci.* **23** (2000), no. 10, 897–908.
- [22] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr. 23, American Mathematical Society, Providence, 1968.
- [23] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
- [24] J.-L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* **20** (1967), 493–519.
- [25] F. Mignot and J.-P. Puel, Inéquations d'évolution paraboliques avec convexes dépendant du temps. Applications aux inéquations quasi variationnelles d'évolution, *Arch. Ration. Mech. Anal.* **64** (1977), no. 1, 59–91.
- [26] F. Miranda and J. F. Rodrigues, On a variational inequality for incompressible non-Newtonian thick flows, in: *Recent Advances in Partial Differential Equations and Applications*, Contemp. Math. 666, American Mathematical Society, Providence (2016), 305–316.
- [27] F. Miranda, J.-F. Rodrigues and L. Santos, On a  $p$ -curl system arising in electromagnetism, *Discrete Contin. Dyn. Syst. Ser. S* **5** (2012), no. 3, 605–629.
- [28] L. Prigozhin, On the Bean critical-state model in superconductivity, *European J. Appl. Math.* **7** (1996), no. 3, 237–247.
- [29] L. Prigozhin, Variational model of sandpile growth, *European J. Appl. Math.* **7** (1996), no. 3, 225–235.
- [30] J.-F. Rodrigues, On the mathematical analysis of thick fluids, *J. Math. Sci. (N.Y.)* **210** (2015), no. 6, 835–848; also published in *Zap. Nauchn. Sem. POMI* **425** (2014), 117–136.
- [31] J. F. Rodrigues and L. Santos, A parabolic quasi-variational inequality arising in a superconductivity model, *Ann. Sc. Norm. Supér. Pisa Cl. Sci. (4)* **29** (2000), no. 1, 153–169.
- [32] J. F. Rodrigues and L. Santos, Quasivariational solutions for first order quasilinear equations with gradient constraint, *Arch. Ration. Mech. Anal.* **205** (2012), no. 2, 493–514.
- [33] T. Roubíček, *Nonlinear Partial Differential Equations with Applications*, 2nd ed., Internat. Ser. Numer. Math. 153, Birkhäuser/Springer, Basel, 2013.
- [34] L. Santos, A diffusion problem with gradient constraint and evolutive Dirichlet condition, *Port. Math.* **48** (1991), no. 4, 441–468.
- [35] L. Santos, Variational problems with non-constant gradient constraints, *Port. Math. (N.S.)* **59** (2002), no. 2, 205–248.
- [36] U. Stefanelli, Nonlocal quasivariational evolution problems, *J. Differential Equations* **229** (2006), no. 1, 204–228.