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ON THE FREE BOUNDARY IN HETEROGENEOUS OBSTACLE-TYPE PROBLEMS WITH TWO PHASES

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Dedicated to Nina N. Uraltseva
on the occasion of her 80th birthday

Abstract. Some properties of the solutions of free boundary problems of obstacle-type with two phases are considered for a class of heterogeneous quasilinear elliptic operators, including the p-Laplacian operator with $1 < p < \infty$. Under a natural nondegeneracy assumption on the interface, when the level set of the change of phase has null Lebesgue measure, a continuous dependence result is proved for the characteristic functions of each phase and sharp estimates are established on the variation of its Lebesgue measure with respect to the L^1-variation of the data, in a rather general framework. For elliptic quasilinear equations whose heterogeneities have appropriate integrable derivatives, it is shown that the characteristic functions of both phases are of bounded variation for the general data with bounded variation. This extends recent results for the obstacle problem and is a first result on the regularity of the free boundary of the heterogeneous two phases problem, which is therefore an interface locally of class C^1 up to a possible singular set of null perimeter.

§1. Introduction

We consider stationary free boundary problems with two phases in the form

$$Au + \lambda_+\chi_{\{u>0\}} - \lambda_-\chi_{\{u<0\}} = f \text{ a.e. in } \Omega \tag{1.1}$$

associated with a quasilinear elliptic operator of p-Laplacian type

$$Au = -\operatorname{div}\big(a(x, \nabla u)\big), \tag{1.2}$$

where Ω is a bounded open connected subset of $\mathbb{R}^n$, $n \geq 2$, and the vector field

$$a(x, \eta)\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$$

is Lipschitz continuous in $x \in \Omega$ and continuous differentiable in $\eta \in \mathbb{R}^n \setminus \{0\}$. Here $f = f(x)$ and $\lambda_\pm = \lambda_\pm(x) \geq 0$ are given bounded functions, and $\chi_{\{u>0\}}$, $\chi_{\{u<0\}}$ denote the characteristic functions of each phase

$$\{u > 0\} = \big\{x \in \Omega : u(x) > 0\big\} \text{ and } \{u < 0\} = \big\{x \in \Omega : u(x) < 0\big\}.$$

In the special linear homogeneous case of the Laplacian $A = -\Delta$ with $f \equiv 0$ and constant $\lambda_\pm > 0$, the whole free boundary

$$\Phi = \Omega \cap \big(\partial\{u > 0\} \cup \partial\{u < 0\}\big) \tag{1.3}$$

has finite $(n-1)$-dimensional Hausdorff measure ($\mathcal{H}^{n-1}(\Phi) < \infty$), as it was shown by Weiss [25], and is locally the union of two C^1-surfaces in the neighborhood of each "branch

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point" (see [17]). In the recent monograph [16], the characterization of free boundary points and the analysis of their properties show that the regular and singular one-phase points $\partial\{u > 0\} \setminus \partial\{u < 0\}$ and $\partial\{u < 0\} \setminus \partial\{u > 0\}$ are locally as in the obstacle problem, while the two-phase free boundary points $\partial\{u > 0\} \cap \partial\{u < 0\}$ may be branch points if $|\nabla u| = 0$, or, by the implicit function theorem, open portions of $C^{1,\alpha}$ graphs where $|\nabla u| > 0$, which have been shown to be in fact locally real-analytic.

However, these results cannot be expected to be true always in the case of nonhomogeneous coefficients and more general operators of the type (1.2). In the general case, problem (1.1) was introduced and treated as a variational inequality by Duvaut and Lions in the framework of temperature control problems regulated by interior heat injection (see [9, Chapter 2]). In fact, (1.1) can be regarded as a model for the control of the interface in the steady-state two-phase Stefan problem:

$$\Phi_0 = \{u = 0\} = \{x \in \Omega \,:\, u(x) = 0\}, \tag{1.4}$$

i.e., of the level set of the melting stationary temperature u separating the liquid phase $\{u > 0\}$ from the solid phase $\{u < 0\}$ (see, for instance, [19] and the references therein).

In general, we may have $\Phi_0 \supsetneq \Phi$ and (1.1) should be regarded as a quasilinear partial differential equation with discontinuous nonlinearities

$$Au \in F(x, u) \text{ a.e. in } \Omega, \tag{1.5}$$

where $F(x, u)\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a monotone decreasing function in u for a.e $x \in \Omega$ (see [7] or [20]).

In §2, we use a natural regularization of the Heaviside function to solve (1.5) with the additional mixed boundary conditions, which allows us to characterize equation (1.5) with the right-hand side $f - \zeta(u) \in F(u)$ with a bounded function $\zeta \in \partial J(u)$ related to the characteristic functions of $\{u > 0\}$ and $\{u < 0\}$. Indeed, ζ is given as the element of the subdifferential in u of the convex functional

$$J(v) = \int_\Omega \Big[\lambda_+(x)\, v^+(x) + \lambda_-(x)\, v^-(x)\Big]\, dx, \tag{1.6}$$

where $v^+ = \max(v, 0)$ and $v^- = (-v)^+$.

Although the classical theory of monotone operators [14] yields the existence, uniqueness, and global continuous dependence of the solution of our problems, using the L^1-theory we can obtain an additional interesting estimate on the phase variations. This extends the remark of [21], obtained for the one obstacle problem, to the two phases problem in the case of nondegenerate interfaces, i.e., essentially when $\Phi = \Phi_0$, for which we provide sufficient conditions on the data $\lambda_\pm$ and f in order to hold that nondegeneracy. These results complement, in §3, the general remarks on the stability of both phases in terms of their characteristic functions under a natural nondegeneracy condition.

Finally, in §4, we give bounded variation estimates on Au, by extending to the two-phase heterogeneous case the results obtained earlier in [4] and [6] for the one obstacle problem. This estimate provides the regularity of the nondegenerate free boundary, which, outside a possible singular set of null perimeter, is locally of class C^1. These results are new even for the homogeneous p-Laplacian operator $(1 < p < \infty)$ and for linear second order partial differential operators with Lipschitz coefficients, and are valid for general bounded data $f, \lambda_\pm$ with bounded variation, with their sum $\lambda_+ + \lambda_-$ positive, continuous, and having integrable derivatives.

§2. Approximation and continuous dependence of the solution

In this paper we suppose the standard structural assumptions, with $1 < p < \infty$, for the operator A given by (1.2), with $a(x,0) = 0$ and

$$\sum_{i,j=1}^{n} \frac{\partial a_i}{\partial \eta_j}(x,\eta)\,\xi_i\xi_j \geq \gamma_0\big(\kappa + |\eta|^2\big)^{\frac{p-2}{2}}\,|\xi|^2, \tag{2.1}$$

$$\sum_{i,j=1}^{n} \left|\frac{\partial a_i}{\partial \eta_j}(x,\eta)\right| \leq \gamma_1\big(\kappa + |\eta|^2\big)^{\frac{p-2}{2}}, \tag{2.2}$$

$$\big|a(x_1,\eta) - a(x_2,\eta)\big| \leq \gamma_2|x_1 - x_2|\big(\kappa + |\eta|^2\big)^{\frac{p-1}{2}} \tag{2.3}$$

for some $\kappa \in [0,1]$ and some positive constants γ_0, γ_1, γ_2, for $x \in \Omega$, $\eta \in \mathbb{R}^n \setminus \{0\}$ and for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. It is well known that this implies the existence of $\gamma > 0$ such that (see [23, Lemma 1])

$$\sum_{i=1}^{n}\big(a_i(x,\eta) - a_i(x,\xi)\big)(\eta_i - \xi_i) \geq \gamma \begin{cases} \big(\kappa + |\eta| + |\xi|\big)^{p-2}\,|\eta - \xi|^2 & \text{if } \ p \leq 2, \\ |\eta - \xi|^p & \text{if } \ p \geq 2. \end{cases} \tag{2.4}$$

Therefore, in particular, we cover the heterogeneous quasilinear operators of the p-Laplacian type when $\kappa = 0$,

$$Au = -\operatorname{div}\Big(M(x)\big(\kappa + |\nabla u|^2\big)^{\frac{p-2}{2}}\,\nabla u\Big),$$

where $M(x)$ may be a Lipschitz continuous positive definite matrix uniformly in $x \in \bar{\Omega}$, and it may also include, for $p = 2$, linear second order operators with variable coefficients $M_{ij}(x)$ in divergence form.

We assume that the boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ is Lipschitz, where the two regular components are such that $\mathcal{H}^{n-1}(\Gamma_D) > 0$, with $\mathcal{H}^{n-1}$ denoting the $(n-1)$-dimensional Hausdorff measure.

For equation (1.1) we consider mixed Dirichlet and Neumann boundary conditions

$$u = h \ \text{ on } \ \Gamma_D \ \text{ and } \ \frac{\partial u}{\partial \nu_A} = a(\nabla u)\cdot\vec{n} = g \ \text{ on } \ \Gamma_N. \tag{2.5}$$

We assume that

$$f \in L^\infty(\Omega), \quad g \in L^\infty(\Gamma_N) \text{ and } h \in W^{1,p}(\Omega), \tag{2.6}$$

$$\lambda_\pm \in L^\infty(\Omega), \quad \lambda_\pm \geq 0 \text{ and } \lambda_+(x) + \lambda_-(x) > 0 \text{ a.e. } x \in \Omega, \tag{2.7}$$

and introduce the functional framework

$$V_h = \big\{v \in W^{1,p}(\Omega) \,:\, v = h \ \text{ on } \ \Gamma_D\big\}, \tag{2.8}$$

$$\langle Au, v\rangle = \int_\Omega a(x,\nabla u)\cdot\nabla v \text{ and } \langle L, v\rangle = \int_\Omega fv + \int_{\Gamma_N} gv, \quad v \in V_0, \tag{2.9}$$

Following [9], our problem may now be formulated as the elliptic variational inequality

$$u \in V_h \,:\, J(v) - J(u) \geq \langle L - Au, v - u\rangle, \quad v \in V_h. \tag{2.10}$$

The standard theory of monotone operators allows us to state that our problem is well posed in the following sense (see, for instance, [14] or [22]).

Proposition 2.1. *Under the assumptions* (2.6)–(2.7)*, there exists a unique solution of* (2.10)*, and there exists* $\zeta \in V_0'$ *such that*

$$\zeta = L - Au \in \partial J(u). \tag{2.11}$$

Moreover, for a sequence of data $h_\eta \to h$ in $W^{1,p}(\Omega)$, $f_\eta \to f$, and $\lambda_{\eta\pm} \to \lambda_\pm$ in $L^{p'}(\Omega)$, $g_\eta \to g$ in $L^{p'}(\Gamma_N)$, for the corresponding solutions u_η of (2.10) *we have $u_\eta \to u$ in $W^{1,p}(\Omega)$ as $\eta \to \infty$.*

Remark 2.1. If in (2.5) the Neumann condition is replaced by a two-phase boundary condition of the type

$$\frac{\partial u}{\partial \nu_A} + \mu_+ \chi_{\{u>0\}} - \mu_- \chi_{\{u<0\}} = g \quad \text{on} \quad \Gamma_N.$$

then (2.10) will also be the variational formulation of this problem if we replace J by J_N defined by

$$J_N(v) = \int_\Omega (\lambda_+ v^+ + \lambda_- v^-) + \int_{\Gamma_N} (\mu_+ v^+ + \mu_- v^-).$$

Although we may expect, from (2.11) and from our departure equation (1.1), that

$$\zeta(x) = \lambda_+(x)\chi_{\{u>0\}}(x) - \lambda_-(x)\chi_{\{u<0\}}(x) \text{ a.e. in } x \in \Omega, \tag{2.12}$$

this characterization may not always hold, in particular when $\mathcal{L}^n(\Phi_0) > 0$, i.e., if the mushy region $\{u = 0\}$ has positive Lebesgue measure.

A first characterization towards (2.12) may be obtained by approximating the solution u of (2.10) by solutions u_ε of the regularized equation

$$Au_\varepsilon + \lambda_+ H_\varepsilon(u_\varepsilon) - \lambda_- H_\varepsilon(-u_\varepsilon) = f \quad \text{in} \quad \Omega \tag{2.13}$$

with the same mixed boundary conditions (2.5). Here $\varepsilon > 0$ and H_ε is the Lipschitz approximation of the Heaviside function

$$H_\varepsilon(t) = 0, \;\; t \le 0, \quad H_\varepsilon(t) = \frac{t}{\varepsilon}, \;\; 0 \le t \le \varepsilon, \quad H_\varepsilon(t) = 1, \;\; t \ge \varepsilon. \tag{2.14}$$

Theorem 2.1. *A unique solution u_ε of* (2.13), (2.5) *is uniformly bounded in $W^{1,p}(\Omega) \cap C^{1,\alpha}(\Omega)$ for some α, $0 < \alpha < 1$, and is such that, as $\varepsilon \to 0$,*

$$u_\varepsilon \to u \quad in \quad W^{1,p}(\Omega) \cap C^{1,\alpha'}(\Omega)$$

for all $\alpha' < \alpha$, where u is the solution of (2.10). *Moreover,*

$$\|\nabla(u - u_\varepsilon)\|^p_{L^p(\Omega)} \le \varepsilon\, C_\gamma \quad \text{if} \;\; p \ge 2, \tag{2.15}$$

$$\|\nabla(u - u_\varepsilon)\|^2_{L^2(\Omega')} \le \varepsilon\, C' \quad \text{if} \;\; 1 < p < 2, \tag{2.16}$$

where $C_\gamma = \frac{1}{\gamma}(\|\lambda_+\|_{L^1(\Omega)} + \|\lambda_-\|_{L^1(\Omega)})$ and $C' > 0$ depends on C_γ, on $\Omega' \subset\subset \Omega$, and on an upper bound Λ'_α of $\|\nabla u_\varepsilon\|_{C^{0,\alpha}(\overline{\Omega'})}$, but not on ε.

Proof. We remark that

$$-\lambda_-(x) \le \zeta_\varepsilon(x) = \lambda_+ H_\varepsilon(u_\varepsilon) - \lambda_- H_\varepsilon(-u_\varepsilon) \le \lambda_+(x) \text{ a.e. } x \in \Omega, \tag{2.17}$$

and therefore $Au_\varepsilon \in L^\infty(\Omega)$ uniformly in $\varepsilon > 0$. Hence, by the general $C^{1,\alpha}$ local regularity under the assumptions (2.1)–(2.3), for any $\Omega' \subset\subset \Omega$ we have (see [12, 8] or [23])

$$\|u_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega'})} \le \Lambda'_\alpha \text{ independent of } \varepsilon > 0.$$

Since, by compactness, there is $u \in V_h$ such that, as $\varepsilon \to 0$ and for all $0 < \alpha' < \alpha$,

$$u_\varepsilon \rightharpoonup u \text{ in } W^{1,p}(\Omega)\text{-weak and in } C^{1,\alpha'}(\overline{\Omega'}),$$
$$H_\varepsilon(u_\varepsilon) \rightharpoonup \chi_+ \text{ and } H_\varepsilon(-u_\varepsilon) \rightharpoonup \chi_- \text{ in } L^\infty(\Omega)\text{-weak*},$$

for some functions χ_+ and χ_- satisfying

$$0 \le \chi_+(x) \le 1 \;\text{ and }\; 0 \le \chi_-(x) \le 1 \text{ a.e. } x \in \Omega.$$

By the uniform convergence of u_ε and the definition (2.14), in the open subsets $\{u>0\}$ and $\{u<0\}$ we have, respectively, $\chi_+ \equiv 1$, $\chi_- \equiv 0$ and $\chi_+ \equiv 0$ and $\chi_- \equiv 1$, because this holds at the interior points of those subsets for $H_\varepsilon(u_\varepsilon)$ and $H_\varepsilon(-u_\varepsilon)$ with ε sufficiently small. Hence,

$$\zeta_\varepsilon \rightharpoonup \lambda_+\chi_+ - \lambda_-\chi_- \text{ in } L^\infty(\Omega)\text{-weak*}.$$

Then, as in [9, Chapter 1], passing to the limit in the variational formulation of (2.13), (2.5)

$$\int_\Omega a(\nabla u_\varepsilon)\cdot\nabla w + \int_\Omega \zeta_\varepsilon w = \int_\Omega fw + \int_{\Gamma_N} gw, \quad w\in V_0, \tag{2.18}$$

we may show that u is a unique solution of (2.10). Then, by (2.10) and (2.11) we may conclude that

$$\zeta = \lambda_+\chi_+ - \lambda_-\chi_- \text{ a.e. in } \Omega, \tag{2.19}$$

by the definition of the subdifferential, and we have

$$\int_\Omega a(\nabla u)\cdot\nabla w + \int_\Omega \zeta w = \int_\Omega fw + \int_{\Gamma_N} gw, \quad w\in V_0. \tag{2.20}$$

Consequently, we obtain

$$\int_\Omega \big[a(\nabla u) - a(\nabla u_\varepsilon)\big]\cdot\nabla(u-u_\varepsilon) = \int_\Omega (\zeta_\varepsilon - \zeta)\,(u-u_\varepsilon). \tag{2.21}$$

In order to estimate the right-hand side, we start with the term corresponding to the positive phase:

$$\begin{aligned}
&\int_\Omega \lambda_+\big[H_\varepsilon(u_\varepsilon)-\chi_+\big]\,(u-u_\varepsilon)\\
&\quad\le \int_{\{u>0\}} \lambda_+\big[H_\varepsilon(u_\varepsilon)-1\big]\,(u-u_\varepsilon) + \int_{\{u=0\}} \lambda_+\big[H_\varepsilon(u_\varepsilon)-\chi_+\big]\,(-u_\varepsilon)\\
&\quad\le \int_{\{u>0\}} \lambda_+\big[H_\varepsilon(u_\varepsilon)-1\big]\,(-u_\varepsilon)\\
&\qquad + \int_{\{u=0\}\cap\{u_\varepsilon<0\}} \lambda_+\chi_+u_\varepsilon + \int_{\{u=0\}\cap\{u_\varepsilon>0\}} \lambda_+\big[H_\varepsilon(u_\varepsilon)-1\big]\,(-u_\varepsilon)\\
&\quad\le \varepsilon\|\lambda_+\|_{L^1(\Omega)}
\end{aligned}$$

by the property $1-H_\varepsilon(t)t\le\varepsilon$ for $t\in\mathbb{R}$, and because, as we have seen,

$$0\le \chi_{\{u>0\}} \le \chi_+ \le 1-\chi_{\{u<0\}} \text{ a.e. in } \Omega. \tag{2.22}$$

Repeating the symmetric argument for the negative phase, we conclude that

$$\int_\Omega (\zeta_\varepsilon-\zeta)\,(u-u_\varepsilon) \le \varepsilon\big(\|\lambda_+\|_{L^1(\Omega)} + \|\lambda_-\|_{L^1(\Omega)}\big)$$

and (2.15) follows immediately from (2.21) and (2.4) with $p\ge 2$. For $1<p<2$, we apply (2.4) in $\Omega'\subset\subset\Omega$ and, using the *a priori* bound for the gradients in Ω', we easily get (2.16) with $C' = (2\Lambda'_\alpha)^{2-p}(\|\lambda_+\|_{L^1(\Omega)} + \|\lambda_-\|_{L^1(\Omega)})/\gamma$. The strong convergence in $W^{1,p}(\Omega)$ is now an easy consequence of those estimates. □

Remark 2.2. By the $C^{1,\alpha}$ estimates up to the boundary (see [13]), if the boundary is of class $C^{1,\gamma}$, for the Dirichlet problem ($\Gamma_N=\varnothing$), if $h\in C^{1,\gamma}(\partial\Omega)$, $0<\gamma<1$, then the solution u belongs to $C^{1,\alpha}(\overline{\Omega})$, $0<\alpha<\gamma$, and, as a consequence, estimate (2.16) is valid in the entire Ω. Analogously, for the Neumann problem ($\Gamma_D=\varnothing$), under certain compatibility conditions on the data (see [1]), we may obtain also $C^{1,\alpha}$-regularity up to the boundary for $g\in C^{0,\gamma}(\partial\Omega)$, $0<\gamma<1$.

Remark 2.3. As a consequence of the proof of Theorem 2.1, the solution u of (2.10) also solves (1.1) in a weak form

$$Au + \lambda_+\chi_+ - \lambda_-\chi_- = f \text{ a.e. in } \Omega, \tag{2.23}$$

with the boundary conditions (2.5) with bounded functions χ_+ and χ_- satisfying (2.22) and $0 \leq \chi_{\{u<0\}} \leq \chi_- \leq 1 - \chi_{\{u>0\}}$ a.e. in Ω.

Remark 2.4. When $\kappa > 0$ in (2.1)–(2.2), we can guarantee that the solution u has integrable second derivatives, namely (see, e.g., [23, Proposition 1])

$$u \in H^2_{\mathrm{loc}}(\Omega) \text{ if } p \geq 2 \text{ and } u \in W^{2,p}_{\mathrm{loc}}(\Omega) \text{ if } p \leq 2.$$

§3. Stability of the phases under nondegeneracy

In general, we cannot preclude a "thick interface" Φ_0, and by (2.22) and Remark 2.3 it is clear that

$$\{0 < \chi_+ < 1\} \cup \{0 < \chi_- < 1\} \subset \{u = 0\} = \Phi_0,$$

where these possible nonempty subsets are defined up to null sets for the Lebesgue measure $\mathcal{L}^n$. Hence, a natural nondegeneracy condition is

$$\mathcal{L}^n(\Phi_0) = \mathrm{meas}\{u = 0\} = 0, \tag{3.1}$$

which is obviously equivalent to

$$\chi_+ = \chi_{\{u>0\}} \text{ and } \chi_- = \chi_{\{u<0\}} \text{ a.e. in } \Omega. \tag{3.2}$$

For the class of differential operators with the property

$$Av = Aw \text{ a.e. in } \{v = w\}, \tag{3.3}$$

it is simple to provide a sufficient condition on the external force f in order to generate the nondegeneracy condition (3.1):

$$f(x) > \lambda_+(x) \text{ or } f(x) < -\lambda_-(x), \text{ a.e. } x \in \Omega. \tag{3.4}$$

Indeed, by (2.23) and (2.19), if u is a solution to the two-phase problem (2.11) with an operator satisfying (3.3), with $\mathrm{meas}\{u = 0\} > 0$, then we have $Au = 0$ a.e. in $\{u = 0\}$, and therefore

$$-\lambda_- \leq \lambda_+\chi_+ - \lambda_-\chi_- = f \leq \lambda_+ \text{ a.e. in } \{u = 0\},$$

which contradicts (3.4).

Therefore we have shown the following interesting result.

Proposition 3.1. *Under the assumptions* (3.3) *and* (3.4)*, the interface* Φ_0 *has zero Lebesgue measure, i.e.,* (3.1) *holds true.*

Remark 3.1. Property (3.3) is associated with the local regularity of the solutions v and w. For operators in the class (2.1)–(2.3) with $\kappa > 0$, for bounded data f, the standard regularity of nonlinear operators yields the integrability of the second order derivatives (see, e.g., [12, 15] or [23]), because the vector field $a(x, \eta)$ is Lipschitz continuous in x, as was recalled in Remark 2.4. Similarly, for $\kappa = 0$ and $p \leq 2$, in particular, for linear operators and the singular heterogeneous p-Laplacian operator, the local regularity still implies condition (3.3). However, for $\kappa = 0$ and $p > 2$, i.e., for the heterogeneous p-Laplacians of degenerate type, the integrability of the second order derivatives fails, but we still have (3.3) by a well known theorem of Stampacchia and, knowing that, we may prove that $a(\cdot, \nabla u(\cdot)) \in (W^{1,p/(p-1)}_{\mathrm{loc}}(\Omega))^n$ by the usual methods of [12, 15] or [23].

Another immediate consequence of (3.1) and (3.2) is

$$\chi_{\{u>0\}} = 1 - \chi_{\{u<0\}} \text{ a.e. } \Omega, \tag{3.5}$$

and the strong approximation of the characteristic functions of both phases follows easily from (2.13):

$$H_\varepsilon(u_\varepsilon) \to \chi_{\{u>0\}} \text{ and } H_\varepsilon(-u_\varepsilon) \to \chi_{\{u<0\}} \text{ in } L^q(\Omega), \quad q < \infty.$$

Indeed, we already know that these convergences hold weakly, and for $q \geq 2$ it suffices to check the convergence of L^q-norms. This follows from

$$\begin{aligned} \int_\Omega \chi_{\{u>0\}} &= \lim_{\varepsilon\to 0} \int_\Omega H_\varepsilon(u_\varepsilon) \geq \limsup_{\varepsilon\to 0} \int_\Omega H_\varepsilon^q(u_\varepsilon) \\ &\geq \liminf_{\varepsilon\to 0} \int_\Omega H_\varepsilon^q(u_\varepsilon) \geq \int_\Omega \chi_{\{u>0\}}^q = \int_\Omega \chi_{\{u>0\}}, \end{aligned} \tag{3.6}$$

and similarly for $H_\varepsilon(-u_\varepsilon)$ towards $\chi_{\{u<0\}}$.

As in the one obstacle problem (see [18] and [21]), these arguments give also a kind of stability of the two phases in terms of their characteristic functions.

Theorem 3.1. *Let $(u_\eta, \chi_{\eta+}, \chi_{\eta-})$ be a solution of (2.23), (2.5) that corresponds to the data f_η, g_η, h_η, $\lambda_{\eta\pm}$ and converging, as in Proposition 2.1, to a solution u of the limit problem (1.1)–(2.5), under condition (3.2). Then*

$$\chi_{\eta+} \to \chi_{\{u>0\}} \text{ and } \chi_{\eta-} \to \chi_{\{u<0\}} \text{ strongly in } L^q(\Omega), \quad q < \infty, \tag{3.7}$$

$$\chi_{\{u_\eta>0\}} \to \chi_{\{u>0\}} \text{ and } \chi_{\{u_\eta<0\}} \to \chi_{\{u<0\}} \text{ strongly in } L^q(\Omega), \quad q < \infty. \tag{3.8}$$

Proof. Let, for some subsequence,

$$\chi_{\eta+} \rightharpoonup \chi_+^* \text{ and } \chi_{\eta-} \rightharpoonup \chi_-^* \text{ in } L^\infty(\Omega)\text{-weakly}^*.$$

Since we know that $\chi_{\eta+} = 0$ in $\{u_\eta < 0\}$, we conclude that

$$0 \leq \chi_+^* \leq 1 - \chi_{\{u<0\}} \text{ a.e. in } \Omega$$

from the convergence

$$0 = \int_\Omega u_\eta^- \chi_{\eta+} \to \int_\Omega u^- \chi_+^* = 0.$$

Since also $\chi_{\eta+} = 1$ in $\{u_\eta > 0\}$, taking an arbitrary measurable set $\mathcal{O} \subset \Omega$, we have

$$\int_{\mathcal{O}} u_\eta^+ \chi_{\eta+} = \int_{\mathcal{O}} u_\eta^+ \to \int_O u^+.$$

On the other hand, since $u_\eta^+ \to u^+$ in $L^q(\Omega)$ for any $q < +\infty$, and

$$\int_{\mathcal{O}} u_\eta^+ \chi_{\eta+} \to \int_{\mathcal{O}} u^+ \chi_+^*,$$

we obtain $\int_{\mathcal{O}} u^+ \chi_+^* = \int_{\mathcal{O}} u^+$ for all $\mathcal{O} \subset \Omega$. Therefore, $\chi_+^* = 1$ a.e. in $\{u > 0\}$, and so

$$0 \leq \chi_{\{u>0\}} \leq \chi_+^*.$$

Using the assumption (3.2), we conclude that

$$\chi_+^* = \chi_{\{u>0\}} = 1 - \chi_{\{u<0\}},$$

and, symmetrically, also

$$\chi_-^* = \chi_{\{u<0\}} = 1 - \chi_{\{u>0\}}.$$

The strong convergences follow as in (3.6) and the same arguments apply by replacing in the above proof $\chi_{\eta+}$ by $\chi_{\{u_\eta>0\}}$ and $\chi_{\eta-}$ by $\chi_{\{u_\eta<0\}}$, completing the results. □

Remark 3.2. We stress that the nondegeneracy condition (3.2) is required only for the limit problem and not for the approximating problems. Therefore, the positive and negative phases have a kind of weak stability in Lebesgue measure if the interface has null measure.

In fact, using the L^1-contraction property of m-accretive operators in Banach spaces (see [3]) and extending the stability property of [5] applied to the one obstacle problem in [18] and [21], we can prove the following estimate.

Theorem 3.2. *Let* (u,ζ) *and* $(\widehat{u},\widehat{\zeta})$ *denote the solutions of* (2.11) *corresponding to the data* (f,g) *and* $(\widehat{f},\widehat{g})$, *i.e., the solutions of* (2.23), (2.5) *with the same* h *and* $\lambda_\pm$ *as in Proposition* 2.1. *Then*

$$\|\zeta-\widehat{\zeta}\|_{L^1(\Omega)} \le \|f-\widehat{f}\|_{L^1(\Omega)} + \|g-\widehat{g}\|_{L^1(\Gamma_N)}. \tag{3.9}$$

If, in addition, both solutions satisfy the nondegeneracy condition (3.1) *and*

$$\lambda_+(x)+\lambda_-(x) \ge \mu > 0 \text{ a.e. } x\in\Omega, \tag{3.10}$$

then

$$\operatorname{meas}\big(\{u>0\}\div\{\widehat{u}>0\}\big) \le \frac{1}{\mu}\big(\|f-\widehat{f}\|_{L^1(\Omega)} + \|g-\widehat{g}\|_{L^1(\Gamma_N)}\big) \tag{3.11}$$

where $\div$ *denotes the symmetric difference of sets,* $B\div D=(B\setminus D)\cup(D\setminus B)$.

Proof. Although estimate (3.9) is well known and is a consequence of the general theory of m-accretive operators in $L^1(\Omega)$ (see, for instance, [26] for a Dirichlet problem, or [1] for a nonlinear Neumann problem), for completeness, we sketch a simple proof for our mixed problem (see also [5] or [21]).

Multiply the difference of equations (2.23) for u and $\widehat{u}$,

$$\zeta-\widehat{\zeta} = f-\widehat{f}-(Au-A\widehat{u}) \text{ a.e. in } \Omega,$$

by the measurable function

$$s(x)=\begin{cases}-1 & \text{on } \{u<\widehat{u}\}\cup\{\zeta<\widehat{\zeta}\},\\ 0 & \text{on } \{u=\widehat{u}\}\cap\{\zeta=\widehat{\zeta}\},\\ 1 & \text{on } \{u>\widehat{u}\}\cup\{\zeta>\widehat{\zeta}\},\end{cases}$$

which satisfies $s\in\sigma(u-\widehat{u})$, where σ denotes the maximal monotone graph of the sign function ($\sigma=\partial r$, $r(t)=|t|$).

Integrating by parts, we obtain

$$\|\zeta-\widehat{\zeta}\|_{L^1(\Omega)} = \int_\Omega(\zeta-\widehat{\zeta})s \le \int_\Omega(f-\widehat{f})s + \int_{\Gamma_N}|g-\widehat{g}| \le \|f-\widehat{f}\|_{L^1(\Omega)} + \|g-\widehat{g}\|_{L^1(\Gamma_N)},$$

because

$$-\int_\Omega(Au-A\widehat{u})s \le \int_{\Gamma_N}|g-\widehat{g}|, \quad \forall\, s\in\sigma(u-\widehat{u}).$$

This inequality follows by using the fact that

$$(Au-A\widehat{u})s = (Au-A\widehat{u})\operatorname{sign}(u-\widehat{u}) \text{ a.e. } x\in\Omega$$

for any $s\in\sigma(u-\widehat{u})$, by (3.3), and considering a smooth approximation $\sigma_\varepsilon(t)\to_{\varepsilon\to 0}$ $\operatorname{sign}(t)$ with the boundary conditions (2.5) for u and $\widehat{u}$ in integration by parts.

Finally, under condition (3.1) we have

$$\zeta=\lambda_+\chi_+-\lambda_-\chi_-$$

with

$$\chi_+=\chi_{\{u>0\}}=1-\chi_{\{u<0\}}=1-\chi_-$$

with similar definitions for $\widehat{\zeta}$ with $\widehat{\chi}_+ = \chi_{\{\widehat{u}>0\}}$ and $\widehat{\chi}_- = \chi_{\{\widehat{u}<0\}} = 1 - \widehat{\chi}_+$.

Taking (3.10) into account, we immediately obtain (3.11) from estimate (3.9) and the following inequality a.e. in Ω:

$$|\zeta - \widehat{\zeta}| = \big|\lambda_+(\chi_+ - \widehat{\chi}_+) - \lambda_-(\chi_- - \widehat{\chi}_-)\big| = \big|(\lambda_+ + \lambda_-)\,(\chi_+ - \widehat{\chi}_+)\big| \geq \mu|\chi_+ - \widehat{\chi}_+|. \quad \square$$

Remark 3.3. Inequality (3.11) can be used to estimate the free boundary stability whenever $\Phi_0 = \{u = 0\}$ and $\widehat{\Phi}_0 = \{\widehat{u} = 0\}$ are nondegenerate and both u and $\widehat{u}$ have a monotonicity property, for instance, if $u_{x_n} = \frac{\partial u}{\partial x_n} \geq 0$ and $\widehat{u}_{x_n} \geq 0$. Hence in a cylinder subdomain $D = \omega \times (-L, L) \subset \Omega$ containing Φ_0 and $\widehat{\Phi}_0$, we may define the following upper semicontinuous functions of $x' \in \omega$:

$$\varphi(x') = \inf\big\{x_n \,:\, u(x', x_n) > 0\big\} \text{ and } \widehat{\varphi}(x') = \inf\big\{x_n \,:\, \widehat{u}(x', x_n) > 0\big\},$$

and from (3.11) we see that

$$\|\varphi - \widehat{\varphi}\|_{L^1(\omega)} = \int_D \big|\chi_{\{u>0\}} - \chi_{\{\widehat{u}>0\}}\big| \leq \frac{1}{\mu}\big(\|f - \widehat{f}\|_{L^1(\Omega)} + \|g - \widehat{g}\|_{L^1(\Gamma_N)}\big) \equiv \delta. \tag{3.12}$$

If, moreover, u and $\widehat{u}$ are such that, for some $\varepsilon > 0$,

$$u_e = \nabla \cdot e \geq 0 \text{ and } \widehat{u}_e \geq 0, \quad \forall\, e \in C_\varepsilon = \big\{x \in \mathbb{R}^n \,:\, x_n > \varepsilon|x'|\big\},$$

i.e., are monotone in some cone with axis e_n and opening $2\arctan(1/\varepsilon)$, then, under assumption (3.1), the free boundaries may be locally given by Lipschitz graphs,

$$\Phi_0 \cap D = \big\{x_n = \varphi(x'),\ x' \in \omega\big\} \text{ and } \widehat{\Phi}_0 \cap D = \big\{x_n = \widehat{\varphi}(x'),\ x' \in \omega\big\}$$

with $|\nabla'\varphi| \leq \varepsilon$ and $|\nabla\widehat{\varphi}| \leq \varepsilon$. Then, arguing as in [18, Theorem 6:5.3, p. 200], using (3.12) and the Gagliardo–Nirenberg interpolation inequality, we may also estimate the Hölder norm:

$$\|\varphi - \widehat{\varphi}\|_{C^{0,\alpha}(\omega)} \leq C_\varepsilon\, \delta^{\frac{1-\alpha}{n}} \text{ for any } 0 \leq \alpha < 1.$$

§4. Regularity of the free boundary

In order to obtain the local boundedness of the $\mathcal{H}^{n-1}$-measure of the essential nondegenerate free boundary Φ_0, we shall require a weak differentiability of the data:

$$f \text{ and } \lambda_\pm \text{ are in } BV_{\mathrm{loc}}(\Omega), \tag{4.1}$$

i.e., are of bounded variation in Ω' for all $\Omega' \subset\subset \Omega$ in the sense that

$$\|\nabla f\|(\Omega') = \sup\bigg\{\sum_{i=1}^{\infty} \int_{\Omega'} f \operatorname{div} \vec{\varphi};\ \vec{\varphi} \in C_c^\infty(\Omega')^n,\ \|\vec{\varphi}\|_\infty \leq 1\bigg\} < \infty.$$

We also require that, for some $\kappa \in [0, 1]$,

$$\sum_{i,j=1}^{n} \bigg|\frac{\partial^2 a_i}{\partial x_i\, \partial x_j}(x, \eta)\bigg| \leq \gamma_3\big(\kappa + |\eta|^2\big)^{\frac{p-1}{2}}, \tag{4.2}$$

$$\sum_{i,j,\ell=1}^{n} \bigg|\frac{\partial^2 a_\ell}{\partial \eta_j\, \partial x_i}(x, \eta)\bigg| \leq \gamma_4\big(\kappa + |\eta|^2\big)^{\frac{p-2}{2}}, \tag{4.3}$$

for positive constants γ_3, γ_4, for a.e. $x \in \Omega$ and all $\eta \in \mathbb{R}^n$.

Theorem 4.1. *Let u be the solution of* (2.10) *under the assumptions of Theorem* 2.1. *If* (4.2) *and* (4.3) *are true, then*

$$Au \in BV_{\mathrm{loc}}(\Omega). \tag{4.4}$$

Proof. We recall from (2.20) and the proof of Theorem 2.1 that u can be approximated by the solution u_ε of (2.13), where now we replace λ_+, λ_- and f by smooth approximations $\lambda_{\varepsilon+}$, $\lambda_{\varepsilon-}$ and f_ε with their gradients bounded in $L^1_{\text{loc}}(\Omega)$ uniformly in ε, by assumption (4.1). Hence, we have

$$Au_\varepsilon = f_\varepsilon - \lambda_{\varepsilon+}H_\varepsilon(u_\varepsilon) + \lambda_{\varepsilon-}H_\varepsilon(-u_\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} f - \lambda_+\chi_+ + \lambda_-\chi_- = Au$$

in $L^q(\Omega)$-weak for all $q<\infty$, and it suffices to show that

$$\int_\omega \left|(Au_\varepsilon)_{x_\ell}\right| \le C_\omega, \quad \ell = 1,\dots,n, \tag{4.5}$$

for an arbitrary $\omega \subset\subset \Omega$ and some constant $C_\omega > 0$ independent of $\varepsilon > 0$.

Consider a cut-off function $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \equiv 1$ in ω and $0 \le \varphi \le 1$ in Ω. Let $\sigma_\delta(t)$, $\delta > 0$, be a smooth approximation to $\operatorname{sign}(t)$, i.e., $|\sigma_\delta(t)| \le 1$, $\sigma_\delta' \ge 0$, $\sigma_\delta(0) = 0$ and $\lim_{\delta\to 0}\sigma_\delta(t) = \operatorname{sign}(t)$. Observe that $\sigma_\delta(u_{\varepsilon x_\ell})\,u_{\varepsilon x_\ell}$ is a nonnegative function and

$$\lim_{\delta\to 0} H_\varepsilon'(\pm u_\varepsilon)\,u_{\varepsilon x_\ell}\,\sigma_\delta(u_{\varepsilon x_\ell}) = \left|\big(H_\varepsilon(\pm u_\varepsilon)\big)_{x_\ell}\right| \text{ a.e. in } \omega.$$

From the approximating equation (2.13) we find

$$(Au_\varepsilon)_{x_\ell} = F_\varepsilon - \left[\lambda_{\varepsilon+}H_\varepsilon'(u_\varepsilon) + \lambda_{\varepsilon-}H_\varepsilon'(-u_\varepsilon)\right]u_{\varepsilon x_\ell}, \tag{4.6}$$

where $F_\varepsilon = f_{\varepsilon x_\ell} - (\lambda_{\varepsilon+})_{x_\ell}H_\varepsilon(u_\varepsilon) + (\lambda_{\varepsilon-})_{x_\ell}H_\varepsilon(-u_\varepsilon)$ is uniformly bounded in $L^1_{\text{loc}}(\Omega)$. Hence, if we prove that

$$\int_\Omega \varphi\,\sigma_\delta(u_{\varepsilon x_\ell})\,(-Au_\varepsilon)_{x_\ell} \le C_\varphi, \quad \ell = 1,\dots,n, \tag{4.7}$$

for some constant $C_\varphi > 0$ independent of ε and δ, we shall obtain (4.5) from the estimate

$$\begin{aligned}
&\int_\Omega \varphi\Big[\lambda_{\varepsilon+}\Big|\big(H_\varepsilon(u_\varepsilon)\big)_{x_\ell}\Big| + \lambda_{\varepsilon-}\Big|\big(H_\varepsilon(-u_\varepsilon)\big)_{x_\ell}\Big|\Big]\\
&\quad= \lim_{\delta\to 0}\int_\Omega \varphi\Big[\lambda_{\varepsilon+}H_\varepsilon'(u_\varepsilon) + \lambda_{\varepsilon-}H_\varepsilon'(-u_\varepsilon)\Big]\,u_{\varepsilon x_\ell}\,\sigma_\delta(u_{\varepsilon x_\ell})\\
&\quad= \lim_{\delta\to 0}\int_\Omega \varphi\,\sigma_\delta(u_{\varepsilon x_\ell})\big[F_\varepsilon - (Au_\varepsilon)_{x_\ell}\big] \le \int_\Omega \varphi|F_\varepsilon| + C_\varphi
\end{aligned}$$

by recalling that $\lambda_{\varepsilon+}$ and $\lambda_{\varepsilon-}$ are nonnegative and uniformly bounded.

Finally, in order to prove the remaining estimate (4.7), we integrate by parts as in [6]:

$$\begin{aligned}
\int_\Omega \varphi\,\sigma_\delta(u_{\varepsilon x_\ell})\,(Au_\varepsilon)_{x_\ell} &= -\int_\Omega \big[a(x,\nabla u_\varepsilon)\big]_{x_\ell}\cdot\nabla\big[\varphi\,\sigma_\delta(u_{\varepsilon x_\ell})\big]\\
&= -\sum_{i=1}^n \int_\Omega \frac{\partial a_i}{\partial x_\ell}(x,\nabla u_\varepsilon)\big(\varphi\,\sigma_\delta(u_{\varepsilon x_\ell})\big)_{x_i}\\
&\qquad - \sum_{i,j=1}^n \int_\Omega \frac{\partial a_i}{\partial \eta_j}\,u_{\varepsilon x_\ell x_j}\,\varphi_{x_i}\,\sigma_\delta(u_{\varepsilon x_\ell})\\
&\qquad - \sum_{i,j=1}^n \int_\Omega \frac{\partial a_i}{\partial \eta_j}\,u_{\varepsilon x_\ell x_j}\,u_{\varepsilon x_\ell x_i}\,\sigma_\delta'(u_{\varepsilon x_\ell})\,\varphi\\
&= J_1 + J_2 + J_3.
\end{aligned} \tag{4.8}$$

Using the structural assumptions, we have

$$\begin{aligned}
J_1 &= \sum_{i=1}^{n} \int_\Omega \Big(\frac{\partial^2 a_i}{\partial x_\ell\, \partial x_i} + \sum_{j=1}^{n} \frac{\partial^2 a_i}{\partial x_\ell\, \partial \eta_j}\, u_{\varepsilon x_j x_i} \Big) \varphi\, \sigma_\delta(u_{\varepsilon x_\ell}) \\
&\le \gamma_3 \int_\Omega \big(\kappa + |\nabla u_\varepsilon|^2\big)^{\frac{p-1}{2}} \varphi + \gamma_4 \int_\Omega \big(\kappa + |\nabla u_\varepsilon|^2\big)^{\frac{p-2}{2}} |D^2 u_\varepsilon|\, \varphi \le \frac{1}{2}\, C_\varphi, \\
J_2 &\le \gamma_1 \int_\Omega \big(\kappa + |\nabla u_\varepsilon|^2\big)^{\frac{p-2}{2}} |D^2 u_\varepsilon|\, |\nabla \varphi| \le \frac{1}{2}\, C_\varphi, \\
J_3 &\le -\gamma_0 \int_\Omega \big(\kappa + |\nabla u_\varepsilon|^2\big)^{\frac{p-2}{2}} |\nabla u_{\varepsilon x_\ell}|^2\, \sigma'_\delta(u_{\varepsilon x_\ell})\, \varphi \le 0,
\end{aligned}$$

where the choice of $C_\varphi > 0$ independently of ε is possible because $|\nabla u_\varepsilon| \in L^\infty_{\mathrm{loc}}(\Omega)$, which bound is independent of κ and ε, and $u_\varepsilon \in H^2_{\mathrm{loc}}(\Omega)$ uniformly in ε when $\kappa > 0$.

Since this last estimate fails in general for $\kappa = 0$, we need to estimate J_1 and J_2 independently of $\varepsilon > 0$, with the help of Lemma 4.1 below, with $\kappa = \varepsilon$, where u_ε is now an approximating solution of (2.13), (2.5) with A regularized by A_ε, with each $a_\varepsilon(x, \eta)$ satisfying (2.1)–(2.3) and (4.2)–(4.3) with $\kappa = \varepsilon > 0$. ☐

Lemma 4.1. *Under the assumptions* (4.1)–(4.3) *and* (2.1)–(2.3) *we have the estimate*

$$\int_{\Omega'} \Big[\big(\kappa + |\nabla u_\varepsilon|^2\big)^{\frac{p-2}{2}} |D^2 u_\varepsilon| \Big]^2 \le C', \quad \forall\, \Omega' \subset\subset \Omega, \tag{4.9}$$

where the constant $C' > 0$ depends on $\|\nabla f\|(\Omega'')$, $\|\nabla \lambda_\pm\|(\Omega')$ and $\|\nabla u_\varepsilon\|_{L^\infty(\Omega'')}$ with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, but is independent of $\kappa \in (0, 1]$ and $\varepsilon > 0$.

Proof. Let $G = G(t)$ be a smooth, odd, monotone nondecreasing function, and let $\varphi \in C_c^\infty(\Omega)$, $\varphi \ge 0$. Since $G(u_{\varepsilon x_\ell})\, u_{\varepsilon x_\ell} \ge 0$, if we multiply (4.6) by $\varphi^2 G(u_{\varepsilon x_\ell})$ and integrate in Ω, then we obtain, as in (4.8),

$$\int_\Omega \big[a(x, \nabla u_\varepsilon)\big]_{x_\ell} \cdot \nabla(\varphi^2 G(u_{\varepsilon x_\ell})) \le \int_\Omega F_\varepsilon\, \varphi^2\, G(u_{\varepsilon x_\ell}). \tag{4.10}$$

Setting $t_\varepsilon = (\kappa + |\nabla u_\varepsilon|^2)^{1/2}$ and developing the left-hand side

$$\sum_{i=1}^{n} \int_\Omega \frac{\partial a_i}{\partial x_\ell} \big(\varphi^2 G(u_{\varepsilon x_\ell})\big)_{x_i} + \sum_{i,j=1}^{n} \int_\Omega \frac{\partial a_i}{\partial \eta_j}\, u_{\varepsilon x_\ell x_j} \big(\varphi^2 G'(u_{\varepsilon x_\ell})\, u_{\varepsilon x_\ell x_i} + 2\, \varphi\, \varphi_{x_i}\, G(u_{\varepsilon x_\ell})\big),$$

we use the structural assumptions (2.1)–(2.2) and (4.2)–(4.3) to obtain, from (4.10), for $\ell = 1, \dots, n$:

$$\begin{aligned}
\gamma_0 \int_\Omega t_\varepsilon^{p-2} |\nabla u_{\varepsilon x_\ell}|^2 \varphi^2\, G'(u_{\varepsilon x_\ell}) &\le \sum_{i,j=1}^{n} \int_\Omega \frac{\partial a_i}{\partial \eta_j}\, u_{\varepsilon x_\ell x_i}\, u_{\varepsilon x_\ell x_j}\, \varphi^2\, G'(u_{\varepsilon x_\ell}) \\
&\le \int_\Omega \Big[F_\varepsilon - \sum_{i=1}^{n} \Big(\frac{\partial^2 a_i}{\partial x_\ell\, \partial x_i} + \sum_{j=1}^{n} \frac{\partial^2 a_i}{\partial x_\ell\, \partial \eta_j}\, u_{\varepsilon x_j x_i} \Big) \Big] \varphi^2\, G(u_{\varepsilon x_\ell}) \\
&\quad - \sum_{i,j=1}^{n} \int_\Omega \frac{\partial a_i}{\partial \eta_j}\, u_{\varepsilon x_\ell x_j}\, 2\, \varphi\, \varphi_{x_i}\, G(u_{\varepsilon x_\ell}) \\
&\le C'_\varphi \int_\Omega \Big[\big(|F_\varepsilon| + \gamma_3\, t_\varepsilon^{p-1}\big) \varphi + (\varphi\, \gamma_4 + \gamma_1)\, t_\varepsilon^{p-2} |D^2 u_\varepsilon| \Big] \varphi |G(t_\varepsilon)|,
\end{aligned} \tag{4.11}$$

because $|G(u_{\varepsilon x_\ell})| \le |G(t_\varepsilon)|$.

For $p \geq 2$, we set $G(t) = t$ and from (4.11) we see that

$$\gamma_0 \int_\Omega t_\varepsilon^{p-2} |D^2 u_\varepsilon|^2 \varphi^2$$
$$\leq n\, C'_\varphi \int_\Omega \varphi^2 \Big[\big(|F_\varepsilon| + \gamma_3\, M_\varphi^{p-1} \big) + (\varphi\, \gamma_4 + \gamma_1)\, t_\varepsilon^{p-2}\, |D^2 u_\varepsilon|\, \varphi \Big] M_\varphi,$$

where $M_\varphi = \|t_\varepsilon\|_{L^\infty(\operatorname{supp}\varphi)}$ may be chosen independent of ε and κ.

By the Cauchy–Schwartz inequality and the monotonicity of t^{p-2}, we have

$$\int_{\Omega'} \big[t_\varepsilon^{p-2} |D^2 u_\varepsilon| \big]^2 \leq M_\varphi^{p-2} \int_\Omega t_\varepsilon^{p-2} |D^2 u_\varepsilon|^2\, \varphi^2$$
$$\leq M_\varphi^{p-1}\, C'' \Big(1 + M_\varphi^{p-1} + \int_\Omega |F_\varepsilon|\, \varphi^2 \Big),$$

provided $\varphi \geq 1$ in Ω', which proves (4.9), because $|F_\varepsilon| \leq |\nabla f_\varepsilon| + |\nabla \lambda_{+\varepsilon}| + |\nabla \lambda_{-\varepsilon}|$ and f, λ_+, and λ_- are locally of bounded variation.

For $1 < p < 2$, we set $G(t) = (\varepsilon + t^2)^{\frac{p-2}{2}}\, t$ and since $G'(t) \geq (p-1)\, (\varepsilon + t^2)^{\frac{p-2}{2}}$, from (4.11) with $s_\varepsilon = (\varepsilon + |u_{\varepsilon x_\ell}|^2)^{1/2} \leq t_\varepsilon$ and $\ell = 1, \ldots, n$ we get

$$\gamma_0 (p-1) \int_\Omega s_\varepsilon^{p-2}\, t_\varepsilon^{p-2} |\nabla u_{\varepsilon x_\ell}|^2 \varphi^2$$
$$\leq C'_\varphi \int_\Omega \Big[\big(|F_\varepsilon| + \gamma_3\, t_\varepsilon^{p-1} \big) \varphi + (\varphi\, \gamma_4 + \gamma_1)\, t_\varepsilon^{p-2}\, |D^2 u_\varepsilon| \Big] \varphi\, t_\varepsilon^{p-1}.$$

Again using the Cauchy–Schwartz inequality and observing that now $s_\varepsilon^{p-2} \geq t_\varepsilon^{p-2}$, we may conclude as before that

$$\int_\Omega \big[t_\varepsilon^{p-2} |D^2 u_\varepsilon| \big]^2 \varphi^2 \leq C^*\, M_\varphi^{p-1} \Big(M_\varphi^{p-1} + \int_\Omega |F_\varepsilon|\, \varphi^2 \Big). \qquad \square$$

As a consequence of equation (1.1), in the nondegenerate interface case, using (3.5), we may write

$$\chi_{\{u>0\}} = \frac{f - Au + \lambda_-}{\lambda_+ + \lambda_-} \text{ a.e. in } \omega, \tag{4.12}$$

where we have introduced the subset

$$\omega = \big\{ x \in \Omega \,:\, (\lambda_+ + \lambda_-)(x) > 0 \big\}. \tag{4.13}$$

Assuming now that

$$\lambda_+ + \lambda_- \in C(\Omega) \cap W^{1,1}_{\mathrm{loc}}(\Omega), \tag{4.14}$$

as an immediate consequence of Theorem 4.1, from (4.12) it follows that the characteristic functions $\chi_{\{u>0\}}$ and $\chi_{\{u<0\}}$ of both phases are locally of bounded variation in ω and, by a well-known theorem of De Giorgi (see, e.g., [11]), this yields the following regularity of the free boundary.

Theorem 4.2. *Under the structural conditions* (2.1)–(2.3), (4.2)–(4.3) *on the heterogeneous operator A and the assumptions* (2.6)–(2.7), (4.1), *and* (4.14), *where the interface is nondegenerate, i.e., if $\mathcal{L}^n(\Phi_0 \cap \omega) = 0$, the free boundary is, up to a set of null perimeter* (*i.e., of $\|\nabla \chi_{\{u>0\}}\|$-measure zero*), *the union of an at most countable family of C^1-hypersurfaces.*

Remark 4.1. As is known from measure theory (see, e.g., [10]), an open set $\mathcal{O} \subset \mathbb{R}^n$ whose characteristic function is locally of bounded variation has a boundary $\partial\mathcal{O}$ with locally with finite perimeter. Its singular component $\Sigma \subset \partial\mathcal{O}$, that is, the subset of points with null upper n-dimensional Lebesgue densities with respect to $\mathcal{O}$ and $\mathbb{R}^n \setminus \mathcal{O}$, has null perimeter (i.e., $\|\nabla\chi_{\mathcal{O}}\|(\Sigma) = 0$) and its essential boundary $\partial_e\mathcal{O} = \partial\mathcal{O} \setminus \Sigma$ has locally finite $(n-1)$-dimensional Hausdorff measure. Although the corresponding regularity for the heterogeneous one obstacle problem was shown in [6], implying that a similar conclusion holds for the free boundary in the one-phase local situation, no concrete conclusion is known at the free boundary branch points for general operators with nonhomogeneous data.

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