



# Bloch-Landau-Zener oscillations in a quasi-periodic potential

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Bloch oscillations and Landau-Zener tunneling are ubiquitous phenomena which are sustained by a band-gap spectrum of a periodic Hamiltonian and can be observed in dynamics of a quantum particle or a wavepacket in a periodic potential under action of a linear force. Such physical setting remains meaningful for aperiodic potentials too, although band-gap structure does not exist anymore. Here we consider the dynamics of non-interacting atoms and Bose-Einstein condensates in a quasiperiodic one-dimensional optical lattice subjected to a weak linear force. Excited states with energies below the mobility edge, and thus localized in space, are considered. We show that the observed oscillatory behavior is enabled by tunneling between the initial state and a state (or several states) located nearby in the coordinate-energy space. The states involved in such Bloch-Landau-Zener oscillations are determined by the selection rule consisting of the condition of their spatial proximity and condition of quasis resonances occurring at avoided crossings of the energy levels. The latter condition is formulated mathematically using the Gershgorin circle theorem. The effect of the inter-atomic interactions on the dynamics can also be predicted on the bases of the developed theory. The reported results can be observed in any physical system allowing for observation of the Bloch oscillations, upon introducing incommensurability in the governing Hamiltonian.

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Bloch oscillations and Landau-Zener tunneling are fundamental phenomena predicted nearly a century ago [1–3]. Nowadays, they are experimentally observed for electrons in solids [4–6], cold atoms [7–9] and Bose-Einstein condensates (BECs) in optical lattices (OL) [10–17], arrays of optical waveguides [18–21], periodic dielectric media [22], optical resonators [23], exciton-polaritons in microcavities [24], plasmonics [25], acoustic fields in water cavities [26] and phononic crystals [27], and in non-Hermitian systems [28,29] (see also reviews [30,31]). In virtue of the ubiquity of these phenomena and possibilities of modifying experimental settings toward including aperiodic potentials, further attention was paid to effect of randomness on Bloch oscillations [32–35], as well as to dynamics of particles in deterministic aperiodic media [36–42]. Despite previous studies, the known results for tilted quasiperiodic potentials, remain scarce, limited to extended states, and somewhat controversial.

Indeed, for a particle in a deep tilted lattice, where aperiodicity is treated as a perturbation and description relies on usage of the conventional Bloch-band theory, numerical studies of [36] revealed sustained dynamics, while in [38] the oscillations were found damped. In Ref. [39] investigating Harper’s tilted lattice, where Bloch theory is not applicable anymore, fragmentation of oscillations in momentum and in real spaces, as well as sensitivity of the dynamics to the

initial state and to the system parameters, were reported. On the other hand, even though the mobility edge (ME) [43] is known to exist for certain types of quasiperiodic potentials [44–47] and was mentioned in [36,39], in the context of Bloch oscillations, only extended states were addressed. Among the available results we also mention experimental observation of the oscillatory dynamics of a BEC in a weakly disordered quasiperiodic lattice [40], and numerical studies of Bloch oscillations in two-dimensional quasiperiodic lattices [37,41].

In this Letter, we report on the dynamics of atoms and BECs in a one-dimensional tilted quasiperiodic OL in the regime when tight-binding approximation is not applicable. We concentrate on the states having energies below the ME and thus, localized in the coordinate space. In the absence of Bloch-band spectrum, the motion of the atoms, referred to below as Bloch-Landau-Zener (BLZ) oscillations, is sustained by simultaneous tunneling in the coordinate space and interlevel transitions enabled by the tilt [see Fig. 1(a)]. The tunneling events can be viewed as quasis resonances where the resonant modes are chosen according to the selection rule. This rule stems from intersection of Gershgorin circles [48] and from the condition of nonnegligible spatial hopping of the wavefunctions.

To describe BLZ oscillations we consider the dimensionless Gross-Pitaevskii equation for the wavefunction  $\Psi$  (normalized to one),

$$i\partial_t \Psi = H_\alpha \Psi + g|\Psi|^2 \Psi. \quad (1)$$

Here  $H_\alpha = H_0 - \alpha x$ ,  $H_0 = -(1/2)\partial_x^2 + V(x)$  is the linear Hamiltonian in the absence of tilt, and  $\alpha$ ,  $0 < \alpha \ll 1$ , is the strength of the linear force. For definiteness below we

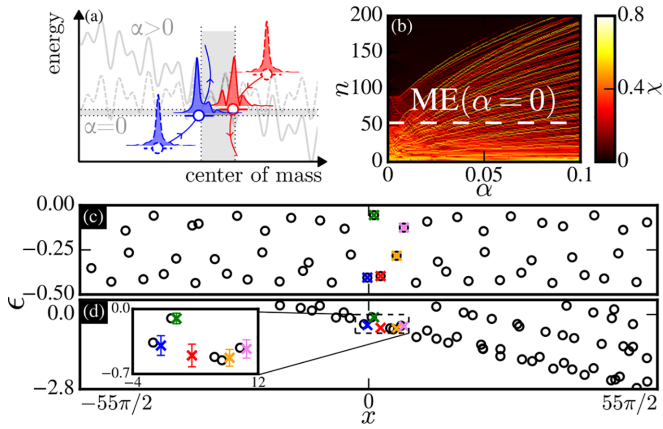


FIG. 1. (a) Schematics of the selection rule. Two states created at  $\alpha = 0$  (dashed lines) when  $\alpha$  increases move in the coordinate-energy space along trajectories shown by arrows. These states do not interact until for certain  $\alpha_1$  they approach each other (solid lines) closer than a distance ensuring nonnegligible hopping and proximity of the energies (intersection of shadowed stripes). At  $\alpha \approx \alpha_1$  the states are in quasiresonance. Gray lines illustrate the potential without (dashed) and with (solid) tilt. (b) IPRs of the eigenstates of  $H_\alpha$ . The dashed line indicates the ME for  $\alpha = 0$ . Energies and centers of masses (empty circles) of states with  $\epsilon_n^{(\alpha)} < \epsilon_{ME}^{(\alpha)}$  for  $\alpha = 0$  and  $\epsilon_{ME}^{(0)} = -0.056$  (c), and  $\alpha = 0.029$  and  $\epsilon_{ME}^{(0.029)} = 0.78$  (d). The colored crosses and diamonds in (c) and (d) represent  $\epsilon_n^{(\alpha)}$  of the states involved in the dynamics shown in Fig. 2. Vertical bars in the inset are the Gershgorin intervals. The parameters are  $v = 1$ ,  $\varphi = (1 + \sqrt{5})/2$ ,  $\theta = 1.23$ , and  $L = 55\pi$ .

concentrate on a quasiperiodic potential  $V(x) = v \cos(2x) + v \cos(2\varphi x + \theta)$ , where  $v$  is the amplitude and  $\varphi$  is an irrational relation between the periods of the sublattices. An arbitrarily chosen  $\theta$  breaks spatial symmetry, and  $g$  characterizes the strength of interatomic interactions. Bichromatic incommensurate potentials with sublattices of comparable amplitudes were studied in Refs. [49–53]. When one of the sublattices has much larger amplitude, the tight-binding approximation becomes applicable and the model is reduced to an Aubry-André-Harper type equation [54–58]. This limit will not be considered here. For cold atoms such potential was produced experimentally in [59].

The spectrum of  $H_0$  with the quasiperiodic potential  $V(x)$  on the whole real axis is of particular complexity [47,60,61]. Meantime a real-world condensate has a finite spatial dimension. Therefore the model can be simplified by considering a BEC in an infinite potential well (see, e.g., [62,63]) whose width  $L$  greatly exceeds the periods of the sublattices:  $L \gg 1$ . This means that  $x \in [-L/2, L/2]$  and zero boundary conditions must be imposed:  $\Psi(\pm L/2, t) = 0$  (for localized states centered far from the boundaries, the assumption that  $L$  is finite is not important [64]). Now our goal can be re-formulated: given an initial ( $t = 0$ ) localized eigenstate of  $H_0$  we explore its dynamics at  $t > 0$  for  $\alpha > 0$ .

To this end we consider the eigenvalue problem  $H_\alpha \phi_n^{(\alpha)}(x) = \epsilon_n^{(\alpha)} \phi_n^{(\alpha)}(x)$ , where  $n$  numbers the orthonormal eigenstates (they are nondegenerate):  $\langle \phi_n^{(\alpha)}, \phi_m^{(\alpha)} \rangle = \delta_{nm}$  [thereafter  $\langle f, g \rangle = \int_{-L/2}^{L/2} f^*(x)g(x)dx$ ], and  $n = 1$  corresponds to the ground state. Starting with the linear case,  $g = 0$ ,

for  $L$  large enough, we can define [50–53,56,65] a ME  $\epsilon_{ME}$  such that all eigenstates of  $H_0$  with  $\epsilon_n^{(0)} < \epsilon_{ME}$  ( $\epsilon_n^{(0)} > \epsilon_{ME}$ ) are localized (delocalized) [see Fig. 1(b)]. Here the localization means that the characteristic width of the state, measured as  $\chi_n^{-1}$ , where  $\chi_n = \langle |\phi_n^{(0)}|^2, |\phi_n^{(0)}|^2 \rangle$  is the inverse participation ratio (IPR), is much less than the size of the system:  $1/\chi_n \ll L$ .

Let  $N$  be the total number of localized states at  $\alpha = 0$ , i.e.,  $\epsilon_N^{(0)} < \epsilon_{ME} < \epsilon_{N+1}^{(0)}$ . A low-energy wavepacket in such lattice without tilt is given by  $\Psi = \sum_{n=1}^N c_n^{(0)}(t) \phi_n^{(0)}(x)$ . Since at  $\alpha > 0$  each eigenstate undergoes displacement in the coordinate-energy space [see Fig. 1(a)], we consider also the “displaced” basis:  $\phi_\alpha = (\phi_1^{(\alpha)}, \dots, \phi_N^{(\alpha)})^T$  ( $T$  stands for the transpose). Strictly speaking, at  $\alpha > 0$  the ME smears out [see Fig. 1(b)] and a possibility of atomic transfer to extended states may occur. However, for the low-energy states the probability of such events remains negligibly small because of smallness of the hopping integrals between localized and delocalized states. This conclusion is supported by numerical simulations that we have performed, thus justifying the use of the  $N$ -mode approximation for  $\alpha \ll 1$ .

The bases  $\phi_0$  and  $\phi_\alpha$  are connected via the unitary transformation  $\phi_\alpha = S_\alpha \phi_0$ , where the entries of the matrix  $S_\alpha$  are  $S_{mn}^{(\alpha)} = \langle \phi_n^{(0)}, \phi_m^{(\alpha)} \rangle$ . Assuming that initially a state  $\phi_n^{(0)}$  is prepared, our task can be reformulated as determining the states  $\phi_m^{(0)}$  (with  $m \neq n$ ) to which the atoms are transferred in a tilted lattice, and describing the respective evolution.

We characterize the spatial position of a localized state with its center of mass (c.m.) which can be expressed using a basis obtained for any  $\alpha$ :  $X(t) = c_\alpha^\dagger(t) X_\alpha c_\alpha(t)$ , where  $c_\alpha = (c_1^{(\alpha)}, \dots, c_N^{(\alpha)})^T$ , and  $X_\alpha$  is an  $N \times N$  matrix with entries  $X_{mn}^{(\alpha)} = \langle \phi_m^{(\alpha)}, x \phi_n^{(\alpha)} \rangle$ . The diagonal element  $X_{nn}^{(\alpha)}$  can be interpreted as the c.m. of a state  $\phi_n^{(\alpha)}$ . Different eigenstates in quasiperiodic potentials are localized at different lattice sites [51,65]. In Fig. 1 this is illustrated for the unperturbed  $H_0$  [panel (c)] and tilted  $H_\alpha$  [panel (d)] Hamiltonians. At  $\alpha = 0$  one observes a nearly homogeneous distribution over the  $x$  axis, while panel (d) for  $\alpha > 0$  can be viewed as a Wannier-Stark ladder (for the lower states) where unlike in the cases of periodic potentials (see, e.g., [66–69]), states with different energies are localized at different spatial steps.

Turning to the linear dynamics, from Eq. (1) with  $g = 0$ , we obtain  $idc_0/dt = (E_0 - \alpha X_0)c_0$  and  $idc_\alpha/dt = E_\alpha c_\alpha$ , where  $E_\alpha = \text{diag}(\epsilon_1^{(\alpha)}, \dots, \epsilon_N^{(\alpha)})$ , in the unperturbed and tilted bases, respectively. The evolution of  $c_n^{(\alpha)}$  is trivial:  $c_\alpha(t) = \Lambda(t)c_\alpha(0)$ , where  $\Lambda(t) = \text{diag}(e^{-i\epsilon_1^{(\alpha)}t}, \dots, e^{-i\epsilon_N^{(\alpha)}t})$ , and in terms of the initial populations, the c.m. has the form [64]

$$X(t) = c_0^\dagger(0) S_\alpha^\dagger \Lambda^\dagger(t) S_\alpha X_0 S_\alpha^\dagger \Lambda(t) S_\alpha c_0(0). \quad (2)$$

Certain entries in this formula give negligible contributions to the dynamics. Indeed, suppose that at  $t = 0$  only a state  $m$  is excited, i.e.,  $c_n^{(0)}(0) = \delta_{nm}$ . Then,  $c_n^{(\alpha)}(0) = S_{nm}^{(\alpha)}$ , i.e., the amplitudes of the state  $\phi_n^{(\alpha)}$  is determined by the hopping integral  $S_{nm}^{(\alpha)}$ . Most of these integrals are negligible, except those involving states satisfying a spatial proximity condition

$$|\langle \phi_n^{(0)}, \phi_m^{(\alpha)} \rangle| > \Delta_x, \quad (3)$$

where  $\Delta_x \ll 1$  is determined by the desired accuracy. Only such states are excited in the oscillatory motion.

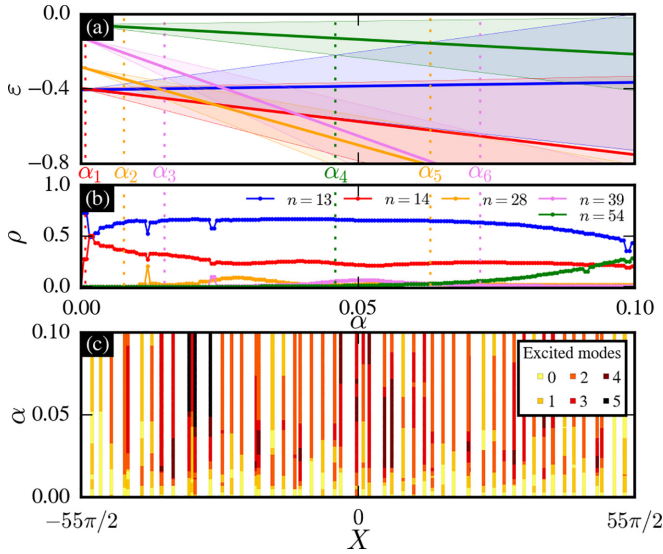


FIG. 2. (a) Centers  $\varepsilon$  (solid lines) of Gershgorin intervals (shaded area) versus tilt. (b) Average populations of excited states obtained from (1) with  $\Psi_0 = \phi_{13}^{(0)}$  for the evolution time  $T = 10^3$ . The vertical dotted lines indicate  $\alpha$  for which the Gershgorin intervals intersect ( $\alpha_1, \dots, \alpha_4$ ) and stop intersecting ( $\alpha_5, \alpha_6$ ). (c) Numbers of states satisfying the selection rule at  $\alpha > 0$  obtained for the states which for  $\alpha = 0$  are centered at  $X_{mn}^{(0)}$  (corresponding to the  $X$  coordinate of colored stripes whose finite widths are used for better visibility). The potential is the same as in Fig. 1.

Furthermore, it follows from (2) that time dependence of the  $X(t)$  is determined by  $\varepsilon_n^{(\alpha)} - \varepsilon_m^{(\alpha)}$ . To express this quantity through the eigenvalues at  $\alpha = 0$ , we consider the diagonal elements of the matrix  $\mathbf{E}_0 - \alpha \mathbf{X}_0$ :  $\varepsilon_n^{(\alpha)} = \varepsilon_n^{(0)} - \alpha X_{nn}^{(0)}$  [for a few states involved in dynamics described below, they are marked by colored crosses in Figs. 1(c) and 1(d)]. Since  $\varepsilon_n^{(\alpha)}$  is an eigenvalue of the matrix  $\mathbf{E}_0 - \alpha \mathbf{X}_0$ , the Gershgorin circle theorem [48] assures that for each  $\varepsilon_n^{(\alpha)}$  there exist  $\varepsilon_m^{(\alpha)}$  such that  $|\varepsilon_n^{(\alpha)} - \varepsilon_m^{(\alpha)}| \leq |R_m^{(\alpha)}|$  where  $R_m^{(\alpha)} = \alpha \sum_{m' \neq m}^N |X_{mm'}^{(0)}|$  are the Gershgorin radii. The subindexes  $m$  and  $n$  in this formula may not coincide because unlike  $\varepsilon_n^{(\alpha)}$  the diagonal elements  $\varepsilon_m^{(\alpha)}$  considered as functions of  $\alpha$  can cross [see Fig. 2(a)]. We refer to such event of crossing as quiresonance. Notice that this quiresonance is essentially different from the resonances induced by driving of the Aubry-André model [58], as well as from recently studied resonances governing behavior of disordered systems in the many-body localized phase [70,71]. Thus,  $\varepsilon_n^{(\alpha)}$  may be found inside a Gershgorin interval (due to reality of the spectrum in our case the circles are squeezed to intervals) centered at  $\varepsilon_m^{(\alpha)}$  originating from the eigenvalue  $\varepsilon_m^{(0)} = \varepsilon_m^{(0)}$ .

The states below the ME are distributed in the space nearly homogeneously [Fig. 1(c)], i.e., the average distance between neighboring states is  $\sim L/N$ , while for their maximal widths one estimates  $1/\chi = 1/\min_m \{\chi_m\}$ . Since  $|X_{mn}^{(0)}| \lesssim 1/\chi$  [64], for  $1/\chi \lesssim L/N$ , valid for our case, only the nearest neighbors contribute to the sum in  $R_m^{(\alpha)}$ . Thus, we obtain the estimate  $|R_m^{(\alpha)}| \lesssim 2\alpha/\chi$ . For the modes in the inset of Fig. 1(d) this yields  $|R_m^{(\alpha)}| \lesssim 0.2$ .

Suppose now, that energies of two eigenstates of  $H_0$ , say  $\varepsilon_m^{(0)}$  and  $\varepsilon_{m'}^{(0)}$ , originate Gershgorin intervals which overlap at

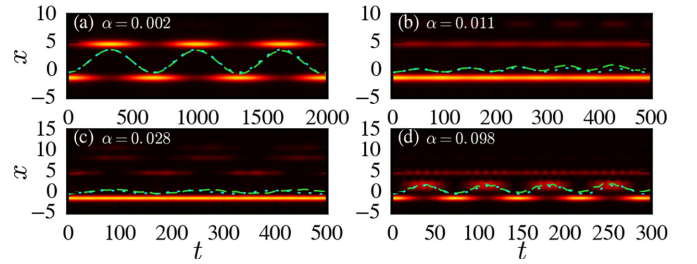


FIG. 3. Evolution of  $|\Psi|^2$  for the initial state  $\Psi_0 = \phi_{13}^{(0)}$  and different field strengths (indicated in the panels). The dashed green and dotted blue line shows evolution of the c.m. obtained from Eq. (1) and the few-mode model. Other parameters are as in Fig. 1. In (b)–(d) only a few periods are shown, while simulations were carried till  $t = 1000$ .

a certain value of  $\alpha$  [see Fig. 1(a)], i.e.,

$$|\varepsilon_m^{(\alpha)} - \varepsilon_{m'}^{(\alpha)}| \leq R_m^{(\alpha)} + R_{m'}^{(\alpha)}. \quad (4)$$

For such states  $|\varepsilon_m^{(\alpha)} - \varepsilon_{m'}^{(\alpha)}| \leq 2(R_m^{(\alpha)} + R_{m'}^{(\alpha)})$ . If  $R_{m,m'}^{(\alpha)}$  are also small enough and spatial hopping of the wavefunctions is nonnegligible, they interact resonantly and can be identified as resonant. Thus, the conditions of spatial proximity (3) and of the overlapping of Gershgorin intervals (4) constitute a selection rule, which allows one, for a given initial state  $m$  to determine a state  $m'$  of the Hamiltonian  $H_0$ , which becomes excited in the tilted quasiperiodic potential. The number of resonant states versus tilt is shown in Fig. 2(c).

The selection rule for sufficiently small  $\alpha$  admits a transparent physical interpretation. Indeed, for strongly localized modes the Gershgorin radii determined by the hopping integrals are small and the quiresonance  $\varepsilon_m^{(\alpha)} = \varepsilon_{m'}^{(\alpha)}$  can be interpreted as approximate equality  $\varepsilon_m^{(0)} - \varepsilon_{m'}^{(0)} \approx \alpha(X_{mm}^{(0)} - X_{m'm'}^{(0)})$ . This means that two states  $m$  and  $m'$  are involved in atomic exchange if the work of the linear force necessary for spatial transfer between their c.m. is approximately equal to the energy difference between the states. The described dynamics corresponds to simultaneous interlevel transitions in the energy space and tunneling in the coordinate space.

Turning to a numerical study of the evolution governed by Eq. (1) with the initial condition  $\Psi(x, t=0) = \Psi_0(x)$ , we characterize participation of a state  $n$  in the dynamics by its average population  $\rho_n = \frac{1}{T} \int_0^T |\langle \phi_n^{(0)}, \Psi \rangle|^2 dt$ , where  $T$  is the total time of the evolution. Figure 2(b) illustrates the results for  $\Psi_0(x) = \phi_{13}^{(0)}(x)$  shown by the blue cross in Fig. 1(c). In close proximity to that state one can find a localized state  $\phi_{14}^{(0)}$  [red cross in Fig. 1(c)]. As a result, already at  $\alpha = \alpha_1 \approx 0.0007$ , tunneling between these states occurs. Tunneling between these states remains for the whole interval  $\alpha \in [0, 0.1]$  [see intersecting blue and red Gershgorin intervals in Fig. 2(a)]. Respectively, in Fig. 2(b) one observes that the mentioned states (blue and red lines) remain populated for all  $\alpha \in [0, 0.1]$ . For  $\alpha < \alpha_2 \approx 0.0079$  (the left yellow dotted line) there are no other excited states satisfying the selection rule, and periodic oscillations [Fig. 3(a)] occur only due to tunneling between the two states. At exact quiresonances corresponding to the intersection of solid lines in Fig. 2(a) when  $\varepsilon_n^{(\alpha)} = \varepsilon_{m'}^{(\alpha)}$ , sharp peaks in the average population of

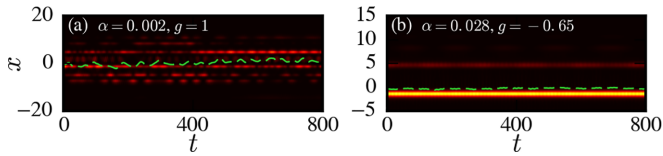


FIG. 4. Evolution of  $|\Psi|^2$  for the initial state  $\Psi_0 = \phi_{13}^{(0)}$  for positive (a) and negative (b) scattering lengths, corresponding to the linear dynamics in Figs. 3(a) and 3(c).

the excited states occur [panel (a)], indicating that the states exchange by the largest amount of atoms.

Upon increase of the linear force, new states may satisfy the selection rule (3), (4), thus becoming excited and involved in the interchange of atoms through BLZ tunneling. In Fig. 2, this is the state  $\phi_{28}^{(0)}$  [yellow crosses in Fig. 1] whose Gershgorin interval starts to overlap with that of the state  $\phi_{13}^{(0)}$  at  $\alpha_2$ . Respectively, for  $\alpha > \alpha_2$  one observes oscillatory dynamics characterized by involvement of three states and thus by two main frequencies of oscillations [see Fig. 3(b)]. Further increase of the linear force results in the involvement of even more states in the evolution. For instance, at  $\alpha = \alpha_3 \approx 0.016$  the state  $\phi_{39}^{(0)}$  becomes involved in the population exchange process [see dynamics in Fig. 3(c)]. Remarkably, the “inverse” process is also possible: when the discs separate upon the increase of  $\alpha$ , the tunneling between the respective states becomes suppressed. For example, this happens for the states  $\phi_{28}^{(0)}$  and  $\phi_{39}^{(0)}$  at  $\alpha_5 \approx 0.062$  and  $\alpha_6 \approx 0.072$  [marked in Fig. 2(b) by the right yellow and pink dotted lines], respectively. Therefore in Fig. 2(b), one observes that these states remain unpopulated for  $\alpha > \alpha_5$  and  $\alpha > \alpha_6$ . The respective BLZ oscillations become “simpler” as shown in Fig. 3(d) where the tunneling to  $\phi_{28}^{(0)}$  and  $\phi_{39}^{(0)}$  is strongly suppressed.

The dynamics in a quasiperiodic potential strongly depends on the choice of the initially populated state. In Fig. 2(c) we show the number of states which are excited in a tilted potential provided that at  $\alpha = 0$  a localized state with a given c.m.  $X_{mn}^{(0)}$  (vertical colored lines) is created. The shown numbers correspond to all excited states satisfying the selection rule (3), (4). Notice that there are states remaining out of resonances (marked as zero excited states), meaning that no oscillations of such states are observed below certain values of  $\alpha$ . At the same time, while increase of the tilt involves more states in the dynamics, this is not the only possible scenario: states may become nonresonant upon increase of  $\alpha$ , because their Gershgorin intervals become separated.

All patterns in Fig. 3 are nondecaying. This is also explained by the selection rule. Indeed, absence of overlap of Gershgorin intervals means a negligibly small tunneling rate which, loosely speaking, excludes excitation of those states at any time. For the same reason for a given  $\alpha$ , the dynamics is well described by a few-mode approximation, i.e., by the reduced column vector  $\mathbf{c}_0$  in which only excited states (of all  $N$  existing) are relevant [the number of states can be

determined from Fig. 2(c)]. For example, for  $\alpha < \alpha_2$  in Fig. 2(a), the vector  $\mathbf{c}_0$  has two components and  $\mathbf{X}_0$  is a  $2 \times 2$  matrix (see [64]). The dynamics illustrating this case is shown in Fig. 3(a). Other panels of this figure show evolution which is well described by three-mode [Figs. 3(b) and 3(d)] and four-mode [Fig. 3(c)] models. Comparing the evolution of the c.m. obtained from the full dynamics (dashed green line) and from the respective few-mode approximations (dotted blue line) reveal excellent agreement in all these cases. Notice that while several states are involved in the dynamics, which thus must be characterized by several frequencies, only one main frequency is observed in all panels. This is explained by the smallness of other frequencies [64] which do not manifest themselves during the evolution time shown in the figure.

The developed theory allows one to predict the effect of weak and even moderate nonlinearity on BLZ oscillations described by (1). A small positive scattering length  $g > 0$ , favoring delocalization, is expected to recruit more adjacent (in the coordinate-energy space) states, thus increasing effective Gershgorin intervals. This naturally results in more complex (compared with the linear limit) dynamics, as shown in Fig. 4(a) [cf. Fig. 3(a)]. At larger scattering lengths, relatively fast dispersion of the initial wavepacket is observed (see [64] for examples). On the other hand, a negative scattering length ( $g < 0$ ) results in stronger localization, and thus, in smaller effective Gershgorin radii leading to suppression of the tunneling and creation of weakly oscillating matter solitons. This is illustrated in Fig. 4(b) where barely visible oscillations of the c.m. are displayed [cf. Fig. 3(c)]. At larger intensities of the attractive two-body interactions, a nearly static soliton is created (not shown here).

To conclude we have described sustained Bloch-Landau-Zener oscillations of atoms and BECs in tilted quasiperiodic lattices. We considered states having energies below the mobility edge. The oscillatory motion occurs due to tunneling of atoms between an initial localized state and another localized state (or states) satisfying the selection rule. This rule consists of the conditions of proximity of the states in coordinate and in energy spaces, achieved due to action of the applied linear force. Depending on the initial condition and on the magnitude of the tilt of the quasiperiodic potential, one can detect two- or a few-mode dynamics, or even no dynamics at all. The reported results can be observed in any physical systems where Bloch oscillations have been realized, since quasiperiodic potentials can be obtained using essentially the same technical tools as periodic ones.

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